M.Sc. Project Report

CHARACTERISTIC CLASSES

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Contents

Chapter 1

Vector Bundles

In this report, a space will mean a *paracompact, Hausdorff* topological space.

1.1 Definitions and Examples

Definition 1. (Vector Bundle) *A k-vector bundle of rank n over the base space B, where k is either the field of real numbers or that of complex numbers, consists of the following: a map of topological spaces $p : E \to B$, where E is called the total space and B the base space, a k-vector space structure on each fibre $p^{-1}(x)$, such that there exists an open cover $\{U_\alpha\}_\alpha$ of B such that there is a homeomorphism $\phi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times k^n$ such that the following diagram commutes*

$$\begin{array}{ccc} p^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times k^n \\ & \searrow \quad \swarrow & \\ & U_\alpha & \end{array}$$

and for each $x \in B$, the map on fibres $p^{-1}(x) \to \{x\} \times k^n$ is k-linear.

Note that if B is empty then there is (up to unique isomorphism) a unique k-vector bundle on B, and its rank n is not well-defined. If $B \neq \emptyset$, then the rank n is well-defined.

For any two U_α and U_β, we have homeomorphisms $\phi_\alpha, \phi_\beta : p^{-1}(U_\alpha \cap U_\beta) \to U_\alpha \cap U_\beta \times k^n$ which are k-linear on each fibre, so the map $\phi_\alpha \circ \phi_\beta^{-1} : U_\alpha \cap U_\beta \times k^n \to U_\alpha \cap U_\beta \times k^n$ is also a homeomorphism; the map $x \times k^n \to x \times k^n$ depends continuously on x, which means that we have continuous maps $h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, k)$ satisfying the cocycle conditions $h_{\alpha\beta} \circ h_{\beta\gamma} = h_{\alpha\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$. The maps $h_{\alpha\beta}$ are called the transition funcitons of the vector bundle with respect to the trivialization. If $U_\alpha \cap U_\beta = \emptyset$, then we have $h_{\alpha\beta} = I$. If for a given cover U_α there are two trivializations giving two sets of transition functions $h_{\alpha\beta}$ and $h'_{\alpha\beta}$, then there is a set of continuous functions $g_\alpha : U_\alpha \to GL(n,k)$ such that $h_{\alpha\beta} = g_\alpha^{-1} h'_{\alpha\beta} g_\beta$ holds on $U_\alpha \cap U_\beta$.

Definition 2. *Given two vector bundles $E \xrightarrow{\pi} X$ and $E' \xrightarrow{\pi'} X'$, a morphism of vector bundle is a pair of maps $f : X \to X'$ and $\tilde{f} : E \to E'$ such that $\tilde{f}$ is linear on each fibre and $\pi' \circ \tilde{f} = f \circ \pi$.*

Thus vector bundles over a same base space form a category with morphisms described above; an isomorphism in the category of vector bundles over the base space B is called an isomorphism of bundles over B.

Remark. Two vector bundles E and E' over the same base space B are isomorphic if and only if in a common trivialization U_α, there are functions $g_\alpha : U_\alpha \to GL(n,k)$ such that, if we write the transition functions as $h_{\alpha\beta}$ and $h'_{\alpha\beta}$ respectively, then on $U_\alpha \cap U_\beta$, we have the equality $h_{\alpha\beta} = g_\alpha^{-1} h'_{\alpha\beta} g_\beta$. It is immediately seen that a vector bundle, in this way, corresponds to a cohomology class in $\check{H}^1(B; GL(n, \mathscr{O}_B))$ where $\mathscr{O}_B$ is the sheaf of continuous functions on B. Conversely, given a covering U_α of the base space, and functions $h_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n,k)$ which satisfy the cocycle condition (i.e. on $U_\alpha \cap U_\beta \cap U_\gamma$, the equality $h_{\alpha\beta} h_{\beta\gamma} = h_{\alpha\gamma}$), one can construct a vector bundle of rank n with transition function $h_{\alpha\beta}$ as follows: take the disjoint union $\coprod_\alpha U_\alpha \times k^n$ and define an equivalence relation by $(x, v) \sim (x, h_{\alpha\beta}(x)v)$ if $x \in U_\alpha \cap U_\beta$ and $(x, v) \in U_\beta \times k^n$; then the equivalence classes with the natural projection on B is the desired vector bundle. Thus, elements of $\check{H}^1(B, GL(n, \mathscr{O}_B))$ are in one-one correspondence with rank n vector bundles over B.

Example 1.1. The first example is the trivial bundle $B \times \mathbb{R}^n$; a vector bundle over B is called *trivial* if it is isomorphic to the trivial bundle. The trivial line bundle will be written ϵ in the sequel.

Example 1.2. Another most obvious examples might be the tangent and the cotangent bundles of differentiable manifolds. If B happens to be a holomorphic manifold, then the holomorphic tangent bundle is a complex manifold, more so, all the maps in the definition of vector bundle turn out to be, in this case, holomorphic map, to produce what is called a *holomorphic vector bundle*.

Example 1.3. On the projective space $\mathbb{P}^n$, there is a tautological line bundle $\mathscr{O}_{\mathbb{P}^n}(-1)$ defined by $\mathscr{O}_{\mathbb{P}^n}(-1) = \{(l, v) \in \mathbb{P}^n : v \in l\}$. These are always non-trivial. For $\mathbb{RP}^n$, we will denote it by γ_n^1.

Example 1.4. If V and W are vector bundles, then we can produce the vector bundles $V \oplus W$, $\mathrm{Hom}(V, W)$ and $V \otimes W$, whose fibres over x are $V_x \oplus W_x$, $\mathrm{Hom}(V_x, W_x)$

and $V_x \otimes W_x$, respectively. If W happens to be the trivial line bundle, then we call $\mathrm{Hom}(V, W)$ the dual vector bundle of V and write it as V^*. Given a vector bundle E, we can form its wedge product bundle $\Lambda^k(E)$, whose fibre over x is $\Lambda^k(E_x)$.

1.1.1 Metric

A *metric* h on a vector bundle $E \xrightarrow{p} X$, is a smooth section of $E^* \otimes E^*$ such that on each fibre, it is symmetric and positive definite.

Proposition 1.1.1. *If X is paracompact, then there is a metric on E.*

The covering of X by trivializing charts can be refined to a locally finite covering, say U_α. Then on each U_α, I take the pullback of the trivial metric on $U_\alpha \times \mathbb{R}^n$ and call it h_α. If ϕ_α is a partition of unity subordinate to U_α, then define $h = \sum_\alpha \phi_\alpha h_\alpha$. This h is a metric on E. □

Remark. The subset of E which consists of vectors of norm less than or equal to one is a deformation retract of E. And E_0, the subset of nonzero vectors deformation retracts onto the sphere bundle E' of norm one vectors of E by the map $(v, t) \to \frac{v}{t\|v\|+(1-t)}$.

Remark. If E has a metric, and F is a subbundle of E (i.e. a subset which becomes a vector bundle with the induced projection, and whose trivializations come from those of E), then there is a subbundle $F^\perp$ such that $F \oplus F^\perp = E$. $F^\perp$ has the following description: in an open set which trivializes both E (and thus F also), there is an orthonormal frame $s_1 \cdots, s_r$ of F; then the set $s_1, \ldots, s_r$ can be extended to an orthonormal local frame $s_1, \ldots, s_r, s_{r+1}, \ldots, s_n$ of E, where $s_{r+1}, \ldots, s_n$ are taken to be the local frame of $F^\perp$.

Remark. If E has a metric h, then it is isomorphic to its own dual E^*, by the canonical map $E \to E^*$, which sends a vector $v \in E_x$ to the map $E_x \to \mathbb{R}$ defined by $w \mapsto h(v, w)$.

1.2 Classification of Vector bundles

Let $X \xrightarrow{f} Y$ be a map of topological spaces and $E \xrightarrow{\pi} Y$ be a vector bundle. Then consider the bundle $\pi^* E$ on X defined by

$$f^* E = \{(x, e) \in X \times E : \pi(e) = f(x)\}.$$

Indeed, with the projection map to its first component, it becomes a vector bundle on X, with the fibre over x the same as the fibre of E over $f(x)$. Thus we have the commutative diagram :

$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{f}} & E \\ {\scriptstyle \pi'} \downarrow & & \downarrow {\scriptstyle \pi} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where the upper horizontal map is given by $(x, e) \mapsto e$. This bundle is called the pullback bundle of E to X by the map f.

1.2.1 Homotopy Property

We will show the following:

Theorem 1.2.1. *Suppose $p : E \to B$ is a vector bundle and $f_0, f_1 : A \to B$ are two homotopic maps. Also assume that A is paracompact. Then f_0^*E and f_1^*E are isomorphic.*

One considers the pullback of E under $F : A \times I \to B$. The restriction of this to $A \times \{0\}$ and $A \times \{1\}$ are identified with f_0^*E and f_1^*E, respectively. The theorem will thus follow immediately from the following lemma:

Lemma 1.2.2. *If X is paracompact, then the restriction of a vector bundle $E \to X \times I$ to $X \times \{0\}$ and $X \times \{1\}$ are isomorphic.*

Proof. We begin by observations:

1. A vector bundle $p : E \to X \times [a, b]$ is trivial if it is trivial on both $X \times [a, c]$ and $X \times [c, b]$ for some $a < c < b$. To see this, let the two trivializations be written as $h_1 : p^{-1}(X \times [a, c]) \cong X \times [a, c] \times \mathbb{R}^n$ and $h_2 : p^{-1}(X \times [c, b]) \cong X \times [c, b] \times \mathbb{R}^n$. Then if they do not match at c, then we can make them match, by modifying h_2, i.e., composing it with the automorphism of $X \times [c, b] \times \mathbb{R}^n$ which is given by the automorphism $h_1 h_2^{-1}$ of $X \times \{c\} \times \mathbb{R}^n$ on each slice $X \times \{t\} \times \mathbb{R}^n$. Once they agree at c, we get a global trivialization.

2. By compactness of I, for every vector bundle $E \to X \times I$, there exists an open cover $\{U_\alpha\}_\alpha$ of X such that E is trivial over each $U_\alpha \times I$.

From 2. and a standard general topological fact, we can extract a *countable* cover $\{V_i\}_i \geq 1$ of X and a partition of unity $\{\phi_i\}_i$ subordinate to V_i, such that each V_i is a disjoint union of open sets each contained in some U_α. The disjoint union condition asserts that E is trivial over each $V_i \times I$. Let us write $\psi_i = \phi_1 + \cdots + \phi_i$,

$\psi_0 = 0$ and $p_i : E_i \to X_i$ be the restriction of E over the graph of $\psi_i = X_i$. Since E is trivial over $V_i \times I$, the natural projection homeomorphism $X_i \to X_{i-1}$ lifts to a homeomorphism $h_i : E_i \to E_{i-1}$, which is isomorphism on each fibre, and identity outside the inverse image under p_i of the graph of $\psi_i|_{V_i}$ since $X_i \to X_{i-1}$ is identity outside V_i.

Around each x, there are only finitely ϕ_i which are non-zero. So, the infinite composition $h = h_1 h_2 \cdots$ makes sense and is an isomorphism from the restriction of E over $X \times \{1\}$ to the restriction to $X \times \{0\}$. □

Remark. Note that if E is an oriented vector bundle over $X \times I$, then each of the maps of bundles obtained above preserves orientation of fibres. Hence we also get the following: *If two maps $X \to B$ are homotopic, and $p : E \to B$ is an oriented vector bundle, then the pullbacks of E by those two maps are isomorphic via an orientation preserving isomorphism.*

1.2.2 Universal Vector Bundle

Now that the pullback depends only on the homotopy class of the map, for any X and a vector bundle $E \to B$ of rank n, the pullback map $Maps(X, B) \to Vect^n(X)$, where $Vect^n(X)$ is the set of isomorphism classes of vector bundles of rank n on X, factors through the homotopy equivalence classes $[X, Y]$. It is natural to seek after a universal bundle over a particular space, i.e. for each n, a rank n vector bundle $\pi : E \to G_n$ such that for any space X, there is a natural pullback map $[X, G_n] \to Vect^n(X)$ is a bijection. Indeed, we can find one; here is the construction:

The real (complex) Grassmannian manifold $G_n(\mathbb{R}^k)$ ($G_n(\mathbb{C}^k)$) is the manifold whose elements are the real (complex) n-planes of $\mathbb{R}^k$ ($\mathbb{C}^k$, respectively). Thus, if $V_n(\mathbb{R}^k)$ is the manifold whose elements are n-tuples of orthonormal vectors in $\mathbb{R}^k$, then we have the quotient map $V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$, which only identifies two tuples which determine the same subspace. The Grassmannian is a compact, Hausdorff, connected space, and it is a smooth manifold also. The inclusions $\mathbb{R}^k \subset \mathbb{R}^{k+1} \subset \cdots$ give inclusions $G_n(\mathbb{R}^k) \subset G_n(\mathbb{R}^{k+1}) \subset \cdots$. We set $G_n := \bigcup_k G_n(\mathbb{R}^k)$, endowed with the direct limit topology. Its elements are nothing but the n-dimensional subspaces of $\mathbb{R}^\infty$.

On the Grassmannians, there are canonical line bundles:

$$E_n(\mathbb{R}^k) = \{(l, v) \in G_n(\mathbb{R}^k) : v \in l\}.$$

The inclusions $\mathbb{R}^k \subset \mathbb{R}^{k+1} \subset \cdots$ give inclusions $E_n(\mathbb{R}^k) \subset E_n(\mathbb{R}^{k+1}) \subset \cdots$. If we set $E_n = \bigcup_k E_n(\mathbb{R}^k)$, endowed with the direct limit topology, we see that it is a line bundle on the infinite Grassmannian G_n via the natural projection.

For complex Grassmannians and the tautological bundles, the construction is much the same; and we have $G_n(\mathbb{C}^\infty) = \bigcup_k G_n(\mathbb{C}^k)$ with direct limit topology, where $G_n(\mathbb{C}^k)$, a quotient of the Stiefel manifold $V_n(\mathbb{C}^k)$ is a compact, holomorphic manifold of complex dimension $n(n-k)$. Again, the tautological bundle $E_n(\mathbb{C}^\infty)$ is the set of pairs (l, v) where l is a complex n-plane, and $v \in l$. This is a rank n complex vector bundle on $G_n(\mathbb{C}^\infty)$

We are going to show that this is the universal bundle we were after:

Theorem 1.2.3. *The natural pullback map $[X, G_n] \to Vect^n(X)$, $[f] \mapsto f^*E_n$ is a bijection. If we write $Vect^n_{\mathbb{C}}(X)$ for the isomorphism classes of complex vector bundles (i.e. isomorphisms are complex linear), then the natural map $[X, G_n(\mathbb{C}^\infty)] \to Vect^n_{\mathbb{C}}(X)$ is a bijection.*

Note that, in the following proof, we are utilising the assumption that X is paracompact. In the following proof, we will mainly argue for the real case, the proof for the complex case can be obtained from the following by replacing "real" by "complex" whenever needed.

Proof. First, one has to observe that for a real, rank n vector bundle $p : E \to X$, an isomorphism $E \cong f^*(E_n)$ is equivalent to giving a map $g : E \to \mathbb{R}^\infty$ that is an $\mathbb{R}$-linear injection on each fibre. Because, if we start with $f : X \to G_n$ and $E \cong f^*(E_n)$, we find the following commutative diagram:

$$\begin{array}{ccccccc} E & \xrightarrow{\cong} & f^*(E) & \xrightarrow{\tilde{f}} & E_n & \xrightarrow{\pi} & \mathbb{R}^\infty \\ & \searrow^{p} & \downarrow & & \downarrow & & \\ & & X & \xrightarrow[f]{} & G_n & & \end{array}$$

where $\pi(l, v) = v$. Set g to be the composition across the upper row, which is injection on each fibre, since each of its components is so. Conversely, given such a $g : E \to \mathbb{R}^\infty$, we can define $f : X \to G_n$ by declaring $f(x)$ to be the n-dimensional $\mathbb{R}$-subspace $g(p^{-1}(x))$, which clearly yields a commutative diagram as above.

Now suppose $p : E \to X$ is a vector bundle. Let $\{U_\alpha\}_\alpha$ be a trivializing cover of $E \to X$. There is, by paracompactness of X, a countable trivializing cover U_i of $E \to X$ and a partition of unity ϕ_i subordinate to U_i. Let $g_i : p^{-1}(U_i) \to \mathbb{R}^n$ be the composition of $p^{-1}(U_i) \cong U_i \times \mathbb{R}^n \to \mathbb{R}^n$, i.e. of the trivialization and the projection on the second factor. The map $(\phi_i p) g_i$ which takes $v \in p^{-1}(U_i)$ to $\phi_i(p(v)) g_i(v)$ can be looked upon as a map $E \to \mathbb{R}^n$ that is zero outside $p^{-1}(U_i)$. Near each point of X only finitely many ϕ_i's are nonzero, and at least one ϕ_i is nonzero, so these $(\phi_i p)g_i$'s are the coordinates of a map $g : E \to (\mathbb{R}^n)^\infty = \mathbb{R}^\infty$, which, moreover, is an $\mathbb{R}$-linear injection on each fibre. This proves the surjectivity of our correspondence.

Now let there be a vector bundle E with two isomorphisms $E \cong f_0^*(E_n)$ and $f_1^*(E_n)$ for two maps $f_0, f_1 : X \to G_n$, which give two maps $g_0, g_1 : E \to \mathbb{R}^\infty$, both $\mathbb{R}$-linear injection on each fibre. Define homotopy $L : \mathbb{R}^\infty \times I \to \mathbb{R}^\infty$ by $L(x_1, x_2, \cdots, t) = (1-t)(x_1, x_2, \cdots) + t(x_1, 0, x_2, 0, \cdots)$. This is $\mathbb{R}$-linear and injective for each t. Composing this with g_0, we can homotope g_0 to the odd-numbered coordinates. Similarly, we can homotope g_1 to the even numbered coordinates. From the new g_0 to the new g_1, we define the homotopy $g_t = (1-t)g_0 + tg_1$, which is $\mathbb{R}$-linear, and by construction injective on each fibre. So, we have a homotopy $f_t(x) := g_t(p^{-1}(x))$ from f_0 to f_1. This shows the injectivity. □

We now introduce the concept of oriented Grassmannian and establish a similar result as above for oriented bundles (see 2.1). The oriented Grassmannian $\tilde{G}_n(\mathbb{R}^k)$ is the space of oriented n-planes in $\mathbb{R}^k$, that is to say, the quotient of $V_n(\mathbb{R}^k)$ obtained by identifying two n-frames when they determine the same *oriented* subspace of $\mathbb{R}^k$. The map $\tilde{G}_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ which identifies two different orientation of the same n-plane has the following description: Let an element of the manifold $V_n(\mathbb{R}^k)$ be represented by an $n \times k$ matrix, whose, say, the submatrix determined by the first n columns is invertible. Thus the matrix is written as $(A|B)$, where A is invertible, and B is anything. However, under the orientable identification, this is identified to $(I|A^{-1}B)$ if $\det A > 0$, and to $(I'|(I'A)^{-1}B)$, where the $n \times n$ diagonal matrix whose $(1,1)$-th term is -1 and other diagonal entries are 1 is I', if $\det A < 0$. We see that if we vary B, we get disjoint neighbourhoods of the two elements in $\tilde{G}_n(\mathbb{R}^k)$, both of which map homeomorphically to the neighbourhood of the element determined by (A, B) in $G_n(\mathbb{R}^k)$ obtained by varying B. Thus, $\tilde{G}_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ is a double cover. Taking the union as before, and putting the weak topology, we get the double cover $\tilde{G}_n \to G_n$. Over $\tilde{G}_n$ we have the canonical rank n bundle $\tilde{E}_n$, which can be described as the pullback of E_n under the covering projection. We see that this vector bundle comes with a canonical choice of orientation on each fibre, which is locally consistent by covering space property of $\tilde{G}_n$, making $\tilde{E}_n$ an orientable vector bundle. Hence, if $f : X \to G_n$ factors through $\tilde{G}_n$, then f^*E_n is an orientable vector bundle on X.

Conversely, if $f : X \to G_n$ yields an orientable vector bundle, then we will show that f factors through $\tilde{G}_n$. The choice of orientation on each fibre gives a canonical lift of f to $\tilde{G}_n$, by declaring $\tilde{f}(x)$ to be the oriented n-plane with underlying unoriented n-plane the same as $f(x)$ and the orientation same as that of $p^{-1}(x)$. The continuity follows from local consistency condition of orientability of E, the local homeomorphism property of covering spaces and the continuity of

f.

By the remark after Theorem 1.2.1, we see that the pullback map $Maps(X, \tilde{G}_n) \to Vect^n_+(X)$, where $Vect^n_+(X)$ is the set of isomorphism classes of rank n vector bundles over X, factors through the homotopy equivalence classes $[X, \tilde{G}_n]$. It is onto, by the above. We will modify the argument in the proof of Theorem 1.2.3 to show that it is injective as well. To do this, we first observe that, much as in the earlier proof, for an orientable vector bundle $p : E \to B$, giving an isomorphism $E \cong f^*\tilde{E}_n$ preserving orientation is same as giving a map $g : E \to \mathbb{R}^\infty$ which is linear injection on each fibre. For, getting a g from f being clear (exactly similar as before), given g, we can define f by declaring $f(x)$ to be the subspace $g(p^{-1}(x))$ with the orientation coming from $p^{-1}(x)$ (well defined, since g is injection on the fibre $p^{-1}(x)$). Then from g_0 to g_1 (which are induced by two orientation preserving isomorphisms $f_0^*\tilde{E}_n \cong E \cong f_1^*\tilde{E}_n$), as in the proof of the earlier theorem, we have to produce a homotopy H, in such a way that it makes sense to say that $H(x,t) = g_t(p^{-1}(x))$ is really a homotopy taking values in $\tilde{G}_n$. To say that our old method works here, we have to establish the following claim: taking sufficiently small neighbourhood V of $g_t(p^{-1}(x))$ in $\tilde{G}_n$, such that $\pi(V)$ (where $\pi : \tilde{G}_n \to G_n$ is the covering projection) is an open neighbourhood of $\pi(g_t(p^{-1}(x)))$ in G_n which lifts to a union of two disjoint open sets in $\tilde{G}_n$, one being V, another, say V', we can find neighbourhood $U \times (t - \epsilon, t + \epsilon)$ of (x, t), which goes within V under H. If $g_t(p^{-1}(x))$ represents an $n \times \infty$ matrix (whose only finitely many columns are nonzero) A, then we choose a minor, say, for simplicity, determined by the first n columns, is nonzero, then we can take $\pi(V)$ to consist of the equivalent classes of $n \times \infty$ matrices whose that very minor is nonzero, V to be the lift where that minor has the same sign as A, and V' to be the other lift, namely, where that minor has the opposite sign. The claim follows immediate as soon as one notes that this minor is a continuous function of x and t. Hence we have the following theorem:

Theorem 1.2.4. *The natural pullback map $[B, \tilde{G}_n] \to Vect^n_+(B)$ given by $[f] \to f^*\tilde{E}_n$ is a bijection, where $Vect^n_+(B)$ denotes the isomorphism classes of oriented vector bundles on B, and isomorphism means one that preserves orientation.*

Remark. A vector bundle is determined by its transition functions for a trivializing cover. For each k, S^k has a contractible cover of two extended hemispheres, whose intersection is a product $S^{k-1} \times (-\epsilon, \epsilon)$. Thus, orientable, rank n vector bundles can be defined by maps $S^{k-1} \to GL^+(n, \mathbb{R})$, called *clutching functions*, by defining a transition function $S^{k-1} \times (-\epsilon, \epsilon) \to GL^+(n, \mathbb{R})$ that restricts to the clutching function at $S^{k-1} \times \{0\}$ and does not depend on the second coordinate. If there is a homotopy $H : S^{k-1} \times I \to GL(n, \mathbb{R})$, H will act as a clutching function for $S^k \times I$, and the vector bundle it produces restricts to $S^{k-1} \times \{0\}$ and $S^{k-1} \times \{1\}$ as the bundles on X whose clutching functions are H_0 and H_1, thus same in $Vect^n_+(S^k)$. Conversely, start from an isomorphism class of oriented vector bundles. The restriction to S^{k-1} (the equator) of its transition function, which is the restriction of a 1-cocycle with values in the sheaf $GL^+(n, \mathcal{O}_{S^k})$, is defined uniquely up to homotopy, since any two representatives of this cocycle will differ by a 1-coboundary, which comes from a 0-cochain i.e. pair of sections of $GL^+(n, \mathcal{O}_{S^k})$ over the extended hemispheres, but $GL^+(n, \mathbb{R})$ is path connected and hemispheres are contractible, so that this pair of functions are null-homotopic. This proves that the inverse map is well defined.

So, the clutching function construction gives a natural bijection :

$$[S^{k-1}, GL^+(n, \mathbb{R})] \xrightarrow{\simeq} Vect^n_+(S^k).$$

As this only uses the path connectedness of $GL^+(n, \mathbb{R})$, the same kind of argument can be given for the following isomorphism:

$$[S^{k-1}, GL(n, \mathbb{C})] \xrightarrow{\simeq} Vect_{\mathbb{C}}(S^k).$$

From this one infers that $\tilde{G}_n$ is simply connected, since $[S^1, \tilde{G}_n] = Vect^n_+(S^1)$ is trivial, for $[S^0, GL^+(n, \mathbb{R})]$ is trivial by path connectedness of $GL^+(n, \mathbb{R})$. G_n and $\tilde{G}_n$ are path connected, for both $Vect^n(pt.) = [pt., G_n]$ and $Vect^n_+(pt.) = [pt., \tilde{G}_n]$ are trivial. It also follows from the existence of a simply connected double cover that the fundamental group of G_n is $\mathbb{Z}/2$.

1.3 Leray-Hirsch Theorem

Lemma 1.3.1. *Given a fibre bundle $p : E \to B$, and a subspace $A \subset B$ such that (B, A) is k-connected, $(E, p^{-1}(A))$ is also k-connected.*

This is because, given a map $g : (D^i, \partial D^i) \to (E, p^{-1}(A))$, for $i \leq k$, since (B, A) is k-connected, there is a homotopy $f : (D^i, \partial D^i) \times I \to (B, A)$ of the composite $pg = f(-, 0)$ to a map f_1 whose image lies inside A; and the homotopy lifting property of fibre bundles gives a homotopy $g_t : (D^i, \partial D^i) \to (E, p^{-1}(A))$ of g to a map with image in $p^{-1}(A)$.

Theorem 1.3.2. (Leray-Hirsch) *Let R be a commutative ring and $E \xrightarrow{p} B$ be a locally trivial map of spaces with the typical fibre F, with the following conditions:*
1. $H^n(F; R)$ is a free R module of finite rank for each n;
2. There are classes $c_j \in H^{k_j}(E; R)$ such that $i_x^*(c_j)$ form a basis of the R-module $H^*(F_x; R)$ for each fibre $i_x : F_x \hookrightarrow E$.

Then the map $\Phi : H^*(B;R)\otimes_R H^*(F;R) \to H^*(E;R)$, $b\otimes i^*(c_j) \mapsto p^*(b) \smile c_j$ *is an isomorphism, where* $i : F \hookrightarrow E$ *stands for the inclusion of the typical fibre.*

A point to observe is that the map Φ is a *graded* map, i.e. it takes graded parts to graded parts.

Proof. The idea is to first establish the result for B a finite CW complex, then apply the above lemma to extend the result to infinite CW complex base, and ultimately apply CW-approximation to generalize it to arbitrary spaces.

Step1: For zero dimensional CW complex base, fibre bundles are product, so that the result follows trivially. Then we use induction on the dimension of the CW complex. Let B be n-dimensional CW complex, then form B' by deleting from the interior of each n-cell a single point, so that B' deformation retracts to the $(n-1)$-th skeleton of B. We assume the result for B', then. Now consider the following diagram, with $E' = p^{-1}(B')$, and the coefficient ring R understood:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & H^*(B,B')\otimes_R H^*(F) & \longrightarrow & H^*(B)\otimes_R H^*(F) & \longrightarrow & H^*(B')\otimes_R H^*(F) & \longrightarrow \cdots \\ & \downarrow \Phi & & \downarrow \Phi & & \downarrow \Phi & \\ \cdots \longrightarrow & H^*(E,E') & \longrightarrow & H^*(E) & \longrightarrow & H^*(E') & \longrightarrow \cdots \end{array}$$

The middle and right vertical maps are as in the statement of the theorem, and the left map can be defined similarly, since $H^*(B,B') \to H^*(E,E')$ is the pullback map and the cup product is $H^*(E)\otimes H^*(E,E') \to H^*(E,E')$. The commutativity of these two squares are obvious, and as to that of the right square, namely, the one involving the coboundary, the argument is this: start with an element $b\otimes i^*(c_j)$ of $H^*(B')\otimes_R H^*(F)$, and map horizontally to $\delta b\otimes i^*(c_j)$, then vertically down to $p^*(\delta b) \smile c_j$ by definition; whereas if we first mapped vertically, we could get $p^*(b) \smile c_j$, which maps horizontally to $\delta(p^*(b) \smile c_j)$, which is $\delta p^*(b) \smile c_j = p^*(\delta b) \smile c_j$, by $\delta c_j = 0$, as c_j comes from $H^*(E)$.

Now, the two rows are exact, the lower one by long exact sequence of (E,E'), the upper one by the long exact sequence of (B,B') and the fact that $H^i(F)$ are always free. So, we can apply five lemma to complete the first step once we show that the left Φ is an isomorphism, since by induction hypothesis and as B' deformation retracts on the $(n-1)$-th skeleton of B, the right Φ is an isomorphism.

For that, let us label the n-cells as e^n_α and the corresponding points taken out of them to get B' as x_α. Now, x_α has a trivializing open neighbourhood $U_\alpha \subset e^n_\alpha$; write $U = \cup_\alpha U_\alpha$ and $U' = U \cap B'$. By excision, $H^*(B,B') \cong H^*(U,U')$ and $H^*(E,E') \cong H^*(p^{-1}(U), p^{-1}(U'))$. Since e^n_α are pairwise disjoint, E is trivial on U and U', so that step one reduces to proving that the map $\Phi : H^*(U,U')\otimes H^*(F) \to H^*(U\times F, U'\times F)$ is an isomorphism. But that can be seen to be true, by five lemma applied to the above diagram with (B,B') replaced by (U,U'), keeping in mind the fact that U and U' deformation retract to CW complexes of dimension 0 and $(n-1)$, respectively.

Step2. The n-the skeleton of B being written as B^n, let us consider the following diagram:

$$\begin{array}{ccc} H^*(B)\otimes_R H^*(F) & \longrightarrow & H^*(B^n)\otimes_R H^*(F) \\ \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \Phi} \\ H^*(E) & \longrightarrow & H^*(p^{-1}(B^n)) \end{array}$$

(B, B^n) is n-connected by cellular approximation, so is $(E, p^{-1}(B^n))$, by the above lemma. Now look at the i-th graded part of the above diagram (note that in this diagram, maps are also graded) for $i \leq n$, where the lower horizontal map is an isomorphism, the right side Φ is an isomorphism by step 1, the upper horizontal map is also an isomorphism (since all cohomologies of B and B^n in that map are below n). So the left side Φ is also an isomorphism.

Step 3 (CW-approximation). Let $f : X \to B$ be a CW-approximation of B. Note that it induces isomorphism on cohomology. Then the commutative diagram

$$\begin{array}{ccc} f^*(E) & \longrightarrow & E \\ \downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow{f} & B \end{array}$$

induces map of the long exact sequences of the two fibre bundles $f^*(E) \to X$ and $E \to B$, wherefore, by five lemma and Step 2, $f^*(E) \to E$ induces isomorphism on homotopy groups, thus on cohomology also. The classes c_j pullback to classes in $H^*(f^*(E))$, which still form a basis of $H^*(F_y)$ for each fibre F_y over $y \in X$. Thus, the naturality of Φ reduces Step 3 to the case of $f^*(E) \to X$. $\square$

Chapter 2

Characteristic Classes

First, we introduce the concept of orientability of a bundle.

2.1 Orientability

Two ordered bases $\{v_1, \ldots, v_n\}$ and $\{v'_1, \ldots, v'_n\}$ of a real vector space V are said to be equivalent if the matrix (a_{ij}) determined by equations $v_i = \sum a_{ij} v'_j$ has positive determinant, i.e. lies in $GL^+(n, \mathbb{R})$. An orientation of a real vector space V is an equivalence class of ordered bases. Since $GL^+(n, \mathbb{R})$ has index two in $GL(n, \mathbb{R})$, a real vector space can have precisely two orientations. An ordered basis of V defines a unique isomorphism $\sigma : V \to \mathbb{R}^n$, sending the ordered basis to the natural basis of $\mathbb{R}^n$. Now, any matrix $A \in GL^+(n, \mathbb{R})$ can be joined by a path f in $GL^+(n, \mathbb{R})$ to I. The path f determines a homotopy $H : (\mathbb{R}^n, \mathbb{R}^n - 0) \times I \to (\mathbb{R}^n, \mathbb{R}^n - 0)$ given by $H(v, t) = f(t)v$. This shows that A^* as a map $H^n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \to H^n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ is identity. Two equivalent orientations of V differ by an automorphism of $\mathbb{R}^n$ of positive determinant, and the above argument shows that the map $\sigma^* : H^n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \to H^n(V, V - 0; \mathbb{Z})$ depends only on the equivalence class of the ordered basis. This defines a map from the orientations to the generators of $H^n(V, V - 0; \mathbb{Z})$. This map is one-one, since, a reflection in $\mathbb{R}^n$, i.e. a map $(x_1, x_2, \ldots, x_n) \mapsto (-x_1, x_2, \ldots, x_n)$, changes the orientation, and also, it induces a map of degree -1 on the unit sphere, and we have the following commutative diagram, where the vertical equalities are induced by the boundary map and the canonical deformation retract of $\mathbb{R}^n - 0$ onto S^{n-1}:

$$\begin{array}{ccc} \tilde{H}^{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) & \xrightarrow[-1]{} & \tilde{H}^{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \\ \| & & \| \\ H^n(\mathbb{R}^n,\mathbb{R}^n-0;\mathbb{Z}) & \longrightarrow & H^n(\mathbb{R}^n,\mathbb{R}^n-0;\mathbb{Z}) \end{array}$$

In the above discussion, we could as well have replaced $\mathbb{Z}$ by $\mathbb{Z}/2$, while in that case, orientibility would have been trivial.

If $E \xrightarrow{p} X$ is a vector bundle of rank n, then an *orientation* on E is a choice of preferred generators for $H^n(F_x, F_x - 0; R)$ such that, given a point x_0, there should be a local coordinate system N, h, a trivialization $h : N \times \mathbb{R}^n \to p^{-1}(N)$ such that the homomorphisms $v \mapsto h(x, v)$ where $x \in N$, are orientation preserving for a chosen orientation on $\mathbb{R}^n$.

A complex vector bundle, i.e. one with its transition functions $\mathbb{C}$-linear, is always orientable. To see this, consider a complex matrix in $GL(n, \mathbb{C})$; it has an underlying real matrix, which comes by writing each complex entry $(a + ib)$ as a square matrix

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

This is an group inclusion $GL(n, \mathbb{C}) \hookrightarrow GL(2n, \mathbb{R})$, which is continuous, and it gives us the notion of a real determinant of a complex matrix. Note that the way the determinant is defined, the image of $GL(n, \mathbb{C})$ is connected in $\mathbb{R}^\times$ and includes 1, since $GL(n, \mathbb{C})$ is connected. So, it lies in the positive component of $\mathbb{R}^\times$. This means that the transition functions in the real sense of a complex vector bundle have positive real determinant.

Consider the sheaf associated to the locally constant presheaf:

$$U \mapsto H^n(E_U, E'_U; \mathbb{Z}).$$

The Espace Étalé of this sheaf is a covering space of X with fibre over x canonically identified with $H^n(E_x, E_x - 0; \mathbb{Z}) = \tilde{H}^{n-1}(S_x^{n-1}; \mathbb{Z})$. We consider the subset of this cover which consists of the generators of the fibres. This is a two sheeted covering space $\pi : \tilde{X} \to X$. An orientation of E is a global section of $\tilde{X}$. If E is a vector bundle over X, then $\pi^* E$ comes with two different orientations.

$H^n(V, V - 0; R)$ is canonically isomorphic to $\tilde{H}^{n-1}(V - 0; R)$. If E is equipped with a metric, then the pair (E, E_0) deformation retracts to the pair of disc and sphere bundles of E, where E_0 is the complement of the zero section. We write the disc bundle still as E and the sphere bundle as E' in this section. If $h : N \times \mathbb{R}^n \to p^{-1}(N)$ is a trivialization, then we extract a trivialization for the disc and sphere bundles. See that the metric on the right side induces a metric on the left $\langle v, u \rangle_x := \langle hv, hu \rangle_x$. One has an automorphism of $N \times \mathbb{R}^n$: take the natural orthonormal basis $e_1, \cdots e_n$ of $\mathbb{R}^n$; and the Gram-Schmidt orthonormalization with respect to the metric $\langle\cdot,\cdot\rangle_x$ produces orthonormal basis $e_1(x), \cdots e_n(x)$, and $e_i(x)$ vary smoothly with respect to x; so the automorphism takes e_i to $e_i(x)$. This, composed with h, gives a trivialization, which we still call h:

$$h : N \times (D^n, S^{n-1}) \cong (p^{-1}(N), p'^{-1}(N)),$$

where $(p, p') : (E, E') \to B$ is the pair of disc, sphere bundles constructed from E.

In terms of discs and spheres, a description of the local consistency condition can be the following: there should be preferred generators for $\tilde{H}^{n-1}(S^{n-1}_x; R) = H^n(D^n_x, S^{n-1}_x; R)$ for all x, and also one for $\tilde{H}^{n-1}(S^{n-1}; R) = H^n(D^n, S^{n-1}; R)$ such that for a given point, there should exist a local coordinate system N around that point and a trivialization h such that in the trivialization obtained above, the maps

$$\tilde{H}^{n-1}(S^{n-1}_x; R) = H^n(D^n_x, S^{n-1}_x; R) \to H^n(D^n, S^{n-1}; R) = \tilde{H}^{n-1}(S^{n-1}; R)$$

always take preferred generator to preferred generator. This is equivalent to the local consistency condition of vector bundles. To see this, all that is needed is to observe that the automorphism of $\mathbb{R}^n$ for each x considered in the above paragraph fixes orientation locally.

Now let γ be a path on X, so that there is a homotopy $\gamma_t : S^{n-1}_{\gamma(0)} \to B : v \to \gamma(t)$. The inclusion $S^{n-1}_{\gamma(0)} \hookrightarrow E'$, by homotopy lifting property of fibration with the typical fibre a CW complex, admits a lift $\tilde{\gamma}_t : S^{n-1}_{\gamma(0)} \to E'$. Lift means $\tilde{\gamma}_t(S^{n-1}_{\gamma(0)}) \subset S^{n-1}_{\gamma(t)}$. Suppose the vector bundle E is orientable over R. So, $\gamma(0)$ and $\gamma(1)$ can be joined by nice coordinate charts with the above properties; then choose points in the intersection of two consecutive charts lying on the curve. Look at a single chart, say N, h. If the restriction of the curve to this chart whose end points are the said chosen points, be g, consider the map $S^{n-1}_{g(0)} \to S^{n-1}_{g(1)} : v \mapsto (\tilde{g}_1(v))$. If this maps regards orientation in the previous chart, we will show that regards the orientation in this chart also. It is enough to show that the map $S^{n-1}_{g(0)} \to S^{n-1} : v \mapsto h^{-1}(\tilde{g}_1(v))$ regards the orientation under the hypothesis. But this map is homotopic to the map $S^{n-1}_{g(0)} \to S^{n-1} : v \mapsto h^{-1}(\tilde{g}_0(v))$ which regards the orientation - so we are done. Thus, the map $L_\gamma : S^{n-1}_{\gamma(0)} \to S^{n-1}_{\gamma(1)}$ arising from these lifts takes preferred generators to preferred ones. This also means that if γ was a loop at a point $x \in X$, then such a lift L_γ would take preferred generator to itself. If R was $\mathbb{Z}$, it means that the map L^*_γ at the cohomology level would be identity. Note that such lifts L_γ have the property $L_\gamma * L_\delta = L_{\gamma*\delta}$, and depend only on the homotopy class of γ.

For a sphere bundle $p' : E' \to B$ we define the notion of *orientability* over the commutative ring R as follows: for any loop γ based at $x \in B$, any lift $L_\gamma : S^{n-1}_x \to S^{n-1}_x$ should induce identity at the cohomology level. It means that if you choose an x in a path component, and declare some generator of $\tilde{H}^{n-1}(S_x^{n-1}; R)$ as the preferred one, then for any other point y in that path component, choose an arbitrary path γ joining y to x, and the map L_γ^* at the cohomology level does not depend on the choice of γ; so in this way we can declare the image of the preferred generator at x to be the preferred generator at y. We can do this for all path components separately. And in a single path component, thus, given any two points y, z, and any path γ joining them, then map L_γ^* takes preferred generator to preferred ones. Thus what the previous paragraph shows is that if a vector bundle is orientable, its sphere bundle is also orientable. Conversely, for $R = \mathbb{Z}$ or $\mathbb{Z}/2$, if the sphere bundle is orientable, then there is a trivializing chart of the vector bundle $h : N \times \mathbb{R}^n \to p^{-1}(N)$ such that N is path connected and for a point x, the trivialization induces orientation preserving map of spheres $S^{n-1} \to S_x^{n-1}$. For, if one starts from an arbitrary trivialization, either the preferred generator of $\tilde{H}^{n-1}(S^{n-1}; R)$ or its negative goes to the preferred generator of $\tilde{H}^{n-1}(S_x^{n-1}; R)$, so that one can device an automorphism of $\mathbb{R}^n$ to fix it. This means, if you take any other point y and join x to y by a path γ, say, then the identity of $\mathbb{R}^n$ goes to a lift of γ, and that has to respect the orientation, so that the trivialization induces orientation preserving map of spheres $S^{n-1} \to S_y^{n-1}$ for all $y \in U$. If the preferred generator of $H^n(F_x, F_x - 0; R)$ is defined to be the preferred generator of $\tilde{H}^{n-1}(S_x^{n-1}; R)$, then it defines an orientation on the vector bundle in this way.

A disc bundle $(E, E') \to B$, i.e. a pair of fibre bundles with typical fibre pair (D^n, S^{n-1}) is said to be orientable if the sphere subbundle E' is. Note that this is equivalent to requiring that given a loop γ at x, if we lift the homotopy $(D_x^n, S_x^{n-1}) \times I \to B : (a, b) \times t \mapsto \gamma(t)$, via the inclusion into (E, E'), to a homotopy $H : (D_x^n, S_x^{n-1}) \times I \to (E, E')$, then H_1^* should be identity map - this is by homotopy extension property of (D^n, S^{n-1}).

If E is a vector bundle over path connected space X, then the notion of orientation gives a homomorphism $\pi(X) \to \mathbb{Z}/2$ which assigns 1 or 0 to each loop γ according as L_γ preserves orientation or not. As the codomain is abelian, it factors through $H_1(X; \mathbb{Z}/2)$ and by path-connectedness and Universal Coefficient theorem, it is merely an element of $H^1(X; \mathbb{Z}/2)$. This can be a way to look at the Stiefel-Whitney class $w_1(E)$ (see Proposition 2.4.3).

2.2 Stiefel-Whitney and Chern Classes

We will define these classes by the unique properties they have:

Theorem 2.2.1. *To each real vector bundle $E \to B$ we can uniquely assign a sequence of elements $w_i(E) \in H^i(B;\mathbb{Z}/2)$, depending only on the isomorphism class of E, such that*
a) it is functorial i.e., $w_i(f^*E) = f^*(w_i(E))$ for a pullback of E by a map f to B;
b) $w_i(E) = 0$ for $i > dimE$;
c) $w(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$, where $w = 1 + w_1 + w_2 + \cdots \in H^*(B;\mathbb{Z}/2)$;
d) for the tautological bundle $E \to \mathbb{R}\mathrm{P}^\infty$, which is the same as $E_1 \to G_1$, $w_1(E)$ is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty;\mathbb{Z}/2)$.

$w_i(E)$ is called the i-th *Stiefel-Whitney* class of the vector bundle $E \to B$. The sum $w = \sum_i w_i$, with for completeness w_0 assumed to be 1, is called the *total Stiefel-Whitney class*, and the condition 2) is interpreted as $w_n(E_1 \oplus E_2) = \sum_{i+j=n} w_i(E_1) \smile w_j(E_2)$. Similarly, we have the complex analogue of the above theorem:

Theorem 2.2.2. *To each real vector bundle $E \to B$ we can uniquely assign a sequence of elemens $c_i(E) \in H^{2i}(B;\mathbb{Z})$, depending only on the isomorphism class of E, such that*
a) it is functorial i.e., $c_i(f^*E) = f^*(c_i(E))$ for a pullback of E by a map f to B;
b) $c_i(E) = 0$ for $i > dimE$;
c) $c(E_1 \oplus E_2) = w(E_1) \smile w(E_2)$, where $w = 1 + c_1 + c_2 + \cdots \in H^*(B;\mathbb{Z})$;
d) for the tautological bundle $E \to \mathbb{C}\mathrm{P}^\infty$, which is the same as $E_1(\mathbb{C}^\infty) \to G_1(\mathbb{C}^\infty)$, $c_1(E)$ is a generator of $H^2(\mathbb{C}\mathrm{P}^\infty;\mathbb{Z})$.

$c_i(E)$ will be called the i-th *Chern class* of the complex vector bundle $E \to B$. The total Chern class $c = \sum_i$ with c_0 assumed to be 1 is interpreted similarly.

To motivate the Chern classes, for example in case of line bundles, consider the following: For any paracompact manifold base B one writes the constant sheaf with stalks $\mathbb{Z}$ as $\mathbb{Z}_B$, the sheaf of complex valued continuous functions as $\mathscr{O}_B$, and the sheaf of nowhere vanishing continuous functions as $\mathscr{O}_B^*$. Then we have the following exact sequence of sheaves:

$$0 \to \mathbb{Z}_B \to \mathscr{O}_B \to \mathscr{O}_B^* \to 0,$$

where the right side map is given by $\phi \mapsto e^{2\pi i\phi}$. This gives a long exact sequence of the sheaf cohomology, and as $\mathscr{O}_B$ is a fine sheaf (for B is paracompact), its higher cohomologies are zero; so we get isomorphisms induced by the boundary maps:

$$H^1(B, \mathscr{O}_B^*) \tilde{\to} H^2(B, \mathbb{Z}_B).$$

This is functorial as follows: if there is a map $f : B \to N$, we have the following commutative diagram:

$$\begin{array}{ccccc}
H^1(N, \mathscr{O}_N^*) & \longrightarrow & H^1(B, f^{-1}\mathscr{O}_N^*) & \longrightarrow & H^1(B, \mathscr{O}_B^*) \\
\downarrow & & \downarrow & & \downarrow \\
H^2(N, \mathbb{Z}_B) & \longrightarrow & H^2(B, f^{-1}\mathbb{Z}_N) & \xrightarrow{\cong} & H^2(B, \mathbb{Z}_B)
\end{array}$$

where the vertical maps are induced by the boundary maps, the left diagram commutes by the functoriality of the pullback map in sheaf cohomology. The functorial homomorphism $\check{H}^1(X, \mathscr{F}) \to H^1(X, \mathscr{F})$, the latter denoting sheaf cohomology, is an isomorphism for any sheaf $\mathscr{F}$ of abelian groups over X, so that we can identify line bundles on B with elements of the sheaf cohomology groups $H^1(B, \mathscr{O}_B^*)$ (see Remark after Definition 2, Chapter 1). The composition along the upper horizontal row is pullback of line bundles. In particular, we have a functorial isomorphism of the group $H^1(B, \mathscr{O}_B^*)$, which is the group of all line bundles, onto the second singular cohomology. This defines the chern class of line bundles. To see that it indeed does, first observe that it is a functorial cohomology class, so that it depends only on how it behaves on the tautological line bundle on $G_1(\mathbb{C}^\infty) = \mathbb{C}^\infty$. But by functoriality, it is sufficient to look at the class of its pullback to $\mathbb{C}\mathrm{P}^1 \cong S^2$, since the inclusion $\mathbb{C}\mathrm{P}^1 \hookrightarrow \mathbb{C}\mathrm{P}^\infty$ induces isomorphism on the second cohomology by cellular approximation. Exactly as in the remark after Theorem 1.2.4, where one can take the extended hemispheres of S^2 to be the affine patches of $\mathbb{C}\mathrm{P}^1$, transition functions for line bundles on $\mathbb{C}\mathrm{P}^1$ are classified by homotopy classes of maps $S^1 \to \mathbb{C}^*$. As the fundamental group of $\mathbb{C}^*$ is abelian, $[S^1, \mathbb{C}^*]$ is same as $\pi_1(\mathbb{C}^*)$, which has the transition function (looked upon as a clutching function) of the tautological line bundle as a generator.

Proof of Theorem 2.2.1. Associated to a vector bundle $\pi : E \to B$ with fibre $\mathbb{R}^n$ is the projective bundle $P(\pi) : P(E) \to B$, where fibres of $P(E)$ are projectivization of those of E. The topology on $P(E)$ can be obtained by declaring it to be the quotient of the complement of zero section under the action of $\mathbb{R}^*$. We obtain a trivialization $U \times \mathbb{R}\mathrm{P}^{n-1}$ of $P(E)$ out of any trivialization $U \times \mathbb{R}^n$ of E.

As in the proof of Theorem 1.2.3, there is a map $g : E \to \mathbb{R}^\infty$, which is injective on each fibre. By the injectivity condition, it makes sense to projectivize the g to a map $P(g) : P(E) \to \mathbb{R}\mathrm{P}^\infty$. $\mathbb{R}\mathrm{P}^\infty$ has its $\mathbb{Z}/2$-cohomology ring the polynomial ring over $\mathbb{Z}/2$ in a generator α of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$. Put $x = P(g)^*(\alpha) \in H^1(P(E); \mathbb{Z}/2)$. The linear injection $\mathbb{R}^n \to \mathbb{R}^\infty$ for each fibre induces embedding $\mathbb{R}\mathrm{P}^{n-1} \hookrightarrow \mathbb{R}\mathrm{P}^\infty$, which in turn induces isomorphism on cohomology groups below n. Thus, x^i restricts to a generator of $H^i(\mathbb{R}\mathrm{P}^{n-1}; \mathbb{Z}/2)$ for each fibre. The class x is well-defined, since the map g is well defined up to homotopy as in the proof of Theorem 1.2.3. By Leray-Hirsch theorem (Theorem 1.3.2), the ring extension $H^*(B; \mathbb{Z}/2) \to H^*(P(E); \mathbb{Z}/2)$ is finite, and has a basis $\{1, x, \ldots, x^{n-1}\}$. So there is a unique polynomial equation of degree n satisfied by x:

$$x^n + w_1(E)x^{n-1} + \cdots + w_n(E) \cdot 1 = 0$$

for certain cohomology classes $w_i(E) \in H^i(B; \mathbb{Z}/2)$, where multiplication is given by pullback and cup product. Define for consistency $w_0(E) = 1$. This polynomial equation, and hence the classes w_i also, are functorial. For, let $\pi' : E' \to B'$ be the pullback of E by a map $f : B' \to B$, so that there is a map $\tilde{f} : E' \to E$ which is isomorphism on each fibre. Hence g is pulled back to $g\tilde{f} : E' \to \mathbb{R}^\infty$, which is linear injection on each fibre. The pullback of x by $P(\tilde{f})^*$ being denoted by x', we have $P(\tilde{f})^*(w_i(E)x^{n-i}0 = f^*(w_i(E))x'^{n-i}$, so that the polynomial equation above transforms to another polynomial equation $\sum_i f^*(w_i(E))x'^{n-i} = 0$.

Property b) holds by construction of w_i.

Now let us prove the Whitney sum formula. The inclusions $E_i \hookrightarrow E_i \oplus E_2$ induces embeddings $P(E_i) \hookrightarrow P(E_i) \oplus P(E_2)$ for $i = 1, 2$, with $P(E_1) \cap P(E_2)$. Fibre wise, $P(E_1)$ in $P(E_1 \oplus E_2)$ is given by pairs of vectors where the second component is zero, and so on. Thus, writing $U_1 = P(E_1 \oplus E_2) \backslash P(E_1)$ and $U_2 = P(E_1 \oplus E_2) \backslash P(E_2)$, we observe that U_1 deformation retracts onto $P(E_2)$, U_2 deformation retracts onto $P(E_1)$. A map $g : E_1 \oplus E_2 \to \mathbb{R}\mathrm{P}^\infty$ which is a linear injection on fibres, restricts to such maps on E_1, E_2 also, so that the class $x \in H^1(P(E_1 \oplus E_2); \mathbb{Z}/2)$ for $E_1 \oplus E_2$ restricts to classes for E_1 and E_2. If E_1 and E_2 have ranks m and n, respectively, consider the classes $w_1 = \sum_j w_j(E_1)x^{m-i}$ and $w_2 = \sum_j w_j(E_2)x^{n-j}$ in $H^*(P(E_1 \oplus E_2), \mathbb{Z}/2)$, with cup product $w_1 w_2 = \sum_j [\sum\limits_{r+s=j} w_r(E_1)w_s(E_2)]x^{m+n-j}$. Thus, w_1 restricts to zero in $H^m(P(E_1); \mathbb{Z}/2)$, so that w_1 comes from a class in $H^m(P(E_1 \oplus E_2), P(E_1); \mathbb{Z}/2)$, which is isomorphic to $H^m(P(E_1 \oplus E_2), U_2; \mathbb{Z}/2)$, and similarly for w_2 also. The following diagram with implicit coefficient ring $\mathbb{Z}/2$, shows that that $w_1 w_2 = 0$:

$$\begin{array}{ccc}
H^{\cdot}(P(E_1 \oplus E_2), U_2) \times H^n(P(E_1 \oplus E_2), U_1) & \xrightarrow{\smile} & H^{m+n}(P(E_1 \oplus E_2), U_1 \cup U_2) = 0 \\
\downarrow & & \downarrow \\
H^m(P(E_1 \oplus E_2)) \times H^n(P(E_1 \oplus E_2)) & \xrightarrow{\smile} & H^{m+n}(P(E_1 \oplus E_2)
\end{array}$$

This shows that $w_j(E_1 \oplus E_2) = \sum\limits_{r+s=j} w_r(E_1)w_s(E_2)$.

For d), note that $P(\pi)$ in case of the tautological line bundle is a homeomorphism, and as one can take g as $(l, v) \mapsto v$, $P(g)$ is same as $P(\pi)$. So, x is a generator of $H^1(P(E); \mathbb{Z}/2) \cong H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$. The identity $x + w_1(E) \cdot 1 = 0$ implies that $w_1(E)$ is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$.

From the above and Theorem 1.2.3, it follows that once one fixes a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$, for vector bundles that split into direct sum of line bundles, the Whitney sum formula gives the Stiefel-Whitney classes. We will see that the uniqueness follows by the following lemma:

Lemma 2.2.3. (Splitting Principle) *For each vector bundle $\pi : E \to B$ there is a space $F(E)$ and a map $p : F(E) \to B$ such that the pullback p^*E splits as a direct sum of line bundles, and $p^* : H^*(B; R) \to H^*(F(E); R)$, where R is $\mathbb{Z}$ or $\mathbb{Z}/2$ according as E is a complex or real vector bundle, is injective.*

To get a line subbundle, we consider $F_1(E) \to B$ to be the projective bundle $P(E) \to B$, so that the pullback has a canonical line subbundle $\{(l, v) \in P(E) \times E | v \in l\}$. This splits the pullback as a sum $L \oplus L^\perp$, where the orthogonal complement is taken with respect to some metric, which always exists (Proposition 1.1.1). By the Leray-Hirsch theorem (Theorem 1.3.2), the map of rings $p^* : H^*(B; R) \to H^*(F(E); R)$ is injective, since, as we saw above, it can be identified to cupping with 1, which is a basis element of $H^*(F(E); R)$ over $H^*(B; R)$. This can be repeated, namely, to split $L^\perp$ further. □

Proof of Theorem 2.2.2. The above argument, with "complex" replaced by "real", whenever possible, and with integral coefficients, goes well, except that α has to be chosen as a generator of $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$, and that we will write the unique polynomial equation that satisfies x rather as $\sum(-1)^i c_i(E) x^{n-i} = 0$, in order that for the tautological line bundle, the polynomial becomes $x - c_1(E) \cdot 1 = 0$, implying $c_1(E) = \alpha$ without sign troubles. Thus, we have proved Theorem 2.2.2 also. □

Proposition 2.2.4. *Regarding a complex vector bundle $E \to B$ of complex rank n as a real vector bundle of real rank $2n$, we have $w_{2i+1}(E) = 0$ and w_{2i} as the image of $c_i(E)$ under the coefficient homomorphism $H^{2i}(B; \mathbb{Z}) \to H^{2i}(B; \mathbb{Z}/2)$.*

Proof. We will denote the real and complex projectivizations of E by $\mathbb{R}\mathrm{P}(E)$ and $\mathbb{C}\mathrm{P}(E)$, respectively. As in the foregoing, there is a map $g : E \to \mathbb{C}^\infty$, which is $\mathbb{C}$-linear injection on each fibre. Then we have the following commutative diagram, each of whose vertical arrows is a fibre bundle with fibre $\mathbb{R}\mathrm{P}^1$:

$$\begin{array}{ccccc} \mathbb{R}\mathrm{P}^{2n-1} & \longrightarrow & \mathbb{R}\mathrm{P}(E) & \xrightarrow{\mathbb{R}\mathrm{P}(g)} & \mathbb{R}\mathrm{P}^\infty \\ \downarrow & & \downarrow p & & \downarrow \\ \mathbb{C}\mathrm{P}^{n-1} & \longrightarrow & \mathbb{C}\mathrm{P}(E) & \xrightarrow[\mathbb{C}\mathrm{P}(g)]{} & \mathbb{C}\mathrm{P}^\infty \end{array}$$

We apply Leray-Hirsch theorem to the bundle $\mathbb{R}\mathrm{P}^\infty \to \mathbb{C}\mathrm{P}^\infty$, with $\mathbb{Z}/2$ coefficients. If β is a generator of $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$, the $\mathbb{Z}/2$-reduction $\overline{\beta} \in H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}/2)$ pulls back to a generator of $H^2(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$, namely the square α^2 of the generator α of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$. Hence, the $\mathbb{Z}/2$-reduction $\overline{x}_{\mathbb{C}}(E) = \mathbb{C}\mathrm{P}(g)^*(\overline{\beta}) \in H^2(\mathbb{C}\mathrm{P}(E); \mathbb{Z}/2)$ of $x_{\mathbb{C}}(E) \in \mathbb{C}\mathrm{P}(g)^*(\beta)$ pulls back to the square of $x_{\mathbb{R}}(E) = \mathbb{R}\mathrm{P}(g)^*(\alpha) \in H^1(\mathbb{R}\mathrm{P}(E);$

$\mathbb{Z}/2$). Consequently, the $\mathbb{Z}/2$-reduction of the defining relation for the Chern class of E, which is $\sum_i \overline{c}_i(E)\overline{x}_{\mathbb{C}}(E)^{n-i} = 0$ pulls back to the relation $\sum \overline{c}_i(E)x_{\mathbb{R}}(R)^{2n-2i} = 0$. Comparing with the defining relation for the Stiefel-Whitney classes of E, the proposition follows. $\square$

2.3 Thom Isomorphism

A fibre bundle pair is a fibre bundle $p : E \to B$ with fibre F and a subset $E' \subset E$ such that $p : E' \to B$ is also a bundle with fibre $F' \subset F$, and the local trivializations of E' comes from those of E.

Proposition 2.3.1. *Let $p : (E, E') \to B$ be a pair of fibre bundles, with typical fibres F, F', such that $H^n(F, F'; R)$ is free R-modules of finite rank for each n. If there exist classes $c_j \in H^{k_j}(E, E'; R)$ whose restrictions form a basis for $H^*(F, F'; R)$ for each fibre F, F', then $H^*(B; R) \otimes H^*(F, F'; R) \to H^*(E, E'; R)$ is a graded isomorphism.*

Note that in terms of modules, the theorem means that $H^*(E, E'; R)$ is a free modules over $H^*(B; R)$ with basis c_j, where the module structure is induced by $(b, c) \mapsto p^*(b) \smile c$, the cup product being the relative cup product $H^*(E; R) \times H^*(E, E'; R) \to H^*(E, E'; R)$.

Proof. Consider the bundle $\hat{E} \to B$ constructed from E by attaching the mapping cylinder M of the map $E' \to B$ to E by identifying the subspace E' of E and the subset E' in M. Thus, the typical fibre $\hat{F}$ will look like one obtained by attaching CF', the cone on F', to F along the subspace F'. Since B is identified as a subspace of $\hat{E}$ at one end of M, we have

$$H^*(\hat{E}, M; R) \cong H^*(\hat{E} - B, M - B; R) \cong H^*(E, E'; R),$$

the first map given by excision, the second by obvious deformation retracts. The long exact sequence of the triple $(\hat{E}, M, B)$ gives the isomorphism $H^*(\hat{E}, M; R) \cong H^*(\hat{E}, B; R)$ since M deformation retracts on B. All these are $H^*(B; R)$-module isomorphisms. Since the identification of B as a subspace of $\hat{E}$ ensures a section of the map $\hat{E} \to B$, which gives, in turn a splitting in the long exact sequence

$$H^*(\hat{E}; R) \cong H^*(\hat{E}, B; R) \oplus H^*(B; R)$$

as $H^*(B; R)$-modules. Now consider two isomorphisms:

$$H^*(\hat{E}, B; R) \cong H^*(\hat{E}, M; R) \cong H^*(E, E'; R)$$

as noted earlier, and the following one

$$H^i(\hat{F};R) \cong H^i(\hat{F},CF';R) \cong H^i(\hat{F}-pt.,CF'-pt.;R) \cong H^i(F,F';R),$$

for $i > 0$, by the fact that a cone is contractible (the $pt.$ being the vertex of the cone), and excision and the obvious deformation retracts. Hence, if $\hat{c}_j \in H^*(\hat{E};R)$ represent the classes c_j, then $\hat{c}_j$ together with 1 restrict to a basis of $H^*(\hat{F};R)$ over R, and the absolute Leray-Hirsch theorem asserts that the classes $\hat{c}_j$ together with 1 form a basis for $H^*(\hat{E};R)$ over $H^*(B;R)$. Hence, c_j form a basis for $H^*(E,E';R)$ over $H^*(B;R)$. □

For a disc bundle $p : (E,E') \to B$ with typical fibre (D^n, S^{n-1}), we note that all the cohomologies of the fibre pair are free; and only the top dimensional cohomology is non-zero. Hence, if there is a class $u \in H^n(E,E';R)$ which restricts to a generator of $H^n(D^n,S^{n-1};R)$, the above theorem asserts that the map $H^i(B;R) \to H^{i+n}(E,E';R),\ b \mapsto p^*(b) \smile u$ is an isomorphism, which we will call the *Thom isomorphism*, while the class u will be called a *Thom class*.

Let R be either $\mathbb{Z}$ or $\mathbb{Z}/2$, we have the following Lemma:

Lemma 2.3.2. *Let $p : (E,E') \to B$ be a disc bundle i.e. with typical fibre (D^n,S^{n-1}), be orientable over the ring R. Also suppose that the base B is a connected CW complex. Then for each $x \in B$ and $i \leq n$, the restriction map $H^i(E,E';R) \to H^i(D^n_x, S^{n-1}_x;R)$ is an isomorphism.*

It means that if it is orientable, then we consider $\mathbb{Z}$ coefficients, otherwise we use $\mathbb{Z}/2$ coefficients. As we noted in section 2.1, we can choose preferred isomorphism $H^n(D_x,S^{n-1}_x;R) \cong R$ for one point x, and that will give preferred isomorphism for all points. This means, the Lemma would secure a preferred isomorphism $H^n(E,E';R) \cong R$ which restricts to the chosen isomorphism for fibres.

Proof. Step1. First we do the case when B is a finite dimensional CW complex, of dimension k, say. Let $U \subset B$ be the subset obtained by deleting one point from the interior of each k-cell of B; and let $V \subset B$ be the union of the open k-cells of B, so that $B = U \cup V$. For a subspace $A \subset B$, let $(E_A, E'_A) \to A$ be the pair of disc, sphere bundles obtained from E, E' by taking the subspaces projecting onto A. Consider the following Mayer-Vietoris sequence, and from now on the R-coefficient will be implicit:

$$H^n(E,E') \to H^n(E_U,E'_U) \oplus H^n(E_V,E'_V) \xrightarrow{\Psi} H^n(E_{U\cap V},E'_{U\cap V}).$$

The first map is injective by the following argument: by Lemma 1.3.1 $E_{U\cap V}$ to the restriction of E to the union of $(k-1)$-spheres, since $U \cap V$ deformation retracts to the disjoint union of $(k-1)$ spheres; so, by induction, $H^{n-1}(E_{U\cap V}, E'_{U\cap V})$ is zero. Thus, by exactness, $H^n(E, E')$ is isomorphic to $\ker \Psi$. But in the similar way, by induction and Lemma 1.3.1, each of the terms $H^n(E_U, E'_U)$, $H^n(E_V, E'_V)$ and $H^n(E_{U\cap V}, E'_{U\cap V})$ is isomorphic to a product of R's, with one R factor for each path component of the spaces involved, projection onto that R-factor being given by restriction to any fibre in that component. The isomorphisms with copies of R are forced by the isomorphisms of $H^n(D^n_x, S^{n-1}_x)$ with R.

Note that the elements of $\ker \Psi$ are given by pairs $(\alpha, \beta) \in H^n(E_U, E'_U) \oplus H^n(E_V, E'_V)$ having the same restriction to $H^n(E_{U\cap V}, E'_{U\cap V})$. For $x \in U \cap V$, consider the following diagram

$$\begin{array}{ccccc} & \swarrow & H^n(E_U, E'_U) \ni \alpha & \searrow & \\ H^n(D^n_x, S^{n-1}_x) & & \longleftarrow & & H^n(E_{U\cap V}, E'_{U\cap V}) \\ & \nwarrow & H^n(E_V, E'_V) \ni \beta & \nearrow & \end{array}$$

This means that the coordinates in terms of the R-factors as above, of α and of β, for each intersecting component separately, are equal. Now we want to show that all the coordinates of α are equal. Now that B is connected, between any two components of U, one can interpolate finitely many components of U and V, each intersecting the neighbours non trivially. By the above observation, that for intersecting components of U and V, the coordinates are equal, we infer that the coordinates for any two components of U are equal for α. Similarly for β, all the coordinates are equal. It only means that $\ker \Psi$ is a copy of R, restricting isomorphically onto $H^n(D^n_x, S^{n-1}_x)$ for each x.

That $H^i(E, E') = 0$ for $i < n$ follows inductively from the Mayer-Vietoris sequence.

Step2. Suppose B is a connected infinite dimensional CW complex. Consider B^k, the k-th skeleton of B, so that (B, B^k) is k-connected. If the pullback bundle on B^{n+1} is written $(E_{n+1}, E'_{n+1}) \to B^{n+1}$, then by Lemma 1.3.1, both (E, E_{n+1}) and (E', E'_{n+1}) are $(n+1)$-connected, so that $H^i(E, E_{n+1}) = 0$ and $H^i(E', E'_{n+1}) = 0$ for $i \leq n+1$. Then by five lemma, the map $(E_{n+1}, E'_{n+1}) \to (E, E')$ induces isomorphism on cohomology up to n. Let $x \in B$; as B is path connected, there is a curve γ joining x and some point $y \in B^{n+1}$. Then we have the commutative diagram for $i \leq n$:

$$
\begin{array}{ccc}
H^i(E,E') & \longrightarrow & H^i(D^n_x, S^{n-1}_x) \\
\simeq\big\downarrow & \searrow & \simeq\big\downarrow L^*_\gamma \\
H^i(E_{n+1}, E'_{n+1}) & \xrightarrow[\simeq]{} & H^i(D^n_y, S^{n-1}_y)
\end{array}
$$

The left and right vertical maps are isomorphisms, while the lower horizontal map is also an isomorphism by Step 1, so that the slant map is also an isomorphism. The upper horizontal map, thus, turns out to be an isomorphism. □

Now let us generalise the situation: let B be any connected base, so that we have a connected CW-complex X with a weak homotopy equivalence $f : X \to B$. Considering the functorial homotopy long exact sequence of fibre bundles, $f^*E \to E$ and $f^*E' \to E'$ induce isomorphism on homotopy, thus on cohomology also. So, by five lemma, it induces isomorphism on cohomology for the map of pairs $(f^*E, f^*E') \to (E, E')$. For a point x in the image of X, we get an isomorphism of $H^n(E, E'; R)$ via restriction with $H^n(D^n_x, S^{n-1}_x; R)$; and thus by the same method as in Step 2 of the previous Lemma, and if the space B was not connected then by CW-approximation for each component separately, we get the following:

Theorem 2.3.3. *(Thom Isomorphism) For any topological space B, and a disc bundle $p : (E, E') \to B$ with typical fibre (D^n, S^{n-1}), which is orientable over R, we have a Thom class $u \in H^n(E, E'; R)$.*

One notes that, as any sphere bundles is orientable over $R = \mathbb{Z}/2$, the disc bundle always has Thom class in coefficient $\mathbb{Z}/2$.

What Thom isomorphism theorem says is that if a disc bundle is orientable, then it will have a Thom class. Conversely, suppose it has a Thom class, say $u \in H^n(E, E')$, the implicit coefficient ring being $\mathbb{Z}$ of $\mathbb{Z}/2$. Let γ be a loop at x in B, and $H : (D^n_x, S^{n-1}_x) \times I \to E$ be a homotopy that lifts γ to a homotopy equivalence L_γ of (D^n_x, S^{n-1}_x) to itself. The map H_1 of pairs is homotopic to the inclusion map. Then consider the commutative diagram

$$
\begin{array}{ccc}
H^n(E,E') & \xrightarrow{H_1^*} & H^n(D^n_x, S^{n-1}_x) \\
 & \searrow & \big\downarrow L^*_\gamma \\
 & & H^n(D^n_x, S^{n-1}_x)
\end{array}
$$

where the slant map is induced by inclusion, and by the above argument is the same map as H_1^*; so the generator of $H^n(D^n_x, S^{n-1}_x)$ coming from the Thom class goes to the same generator under L^*_γ. So L^*_γ is the identity map. So, the disc bundle is orientable.

2.4 Gysin Sequence and The Euler Class

Let $p : E \to B$ be a vector bundle of rank n. Having had a metric on E, we can talk about its disc and sphere bundles $D(E)$ and $S(E)$, respectively. Then if it is orientable over the implicit coefficient ring (always either $\mathbb{Z}$ or $\mathbb{Z}/2$), we have an isomorphism $\Phi : H^i(B) \to H^{i+n}(D(E), S(E))$ defined by $\Phi(b) = p^*(b) \smile u$ for a Thom class u. Then we have a diagram:

$$\begin{array}{ccccccccccc}
\cdots \longrightarrow & H^i(D(E),S(E)) & \xrightarrow{j^*} & H^i(D(E)) & \longrightarrow & H^i(S(E)) & \longrightarrow & H^{i+1}(D(E),S(E)) & \longrightarrow \cdots \\
& \simeq \uparrow \Phi & & \simeq \uparrow p^* & & \| & & \simeq \uparrow \Phi & \\
\cdots \longrightarrow & H^{i-n}(B) & \xrightarrow{\smile e} & H^i(B) & \longrightarrow & H^i(S(E)) & \longrightarrow & H^{i-n+1}(B) & \longrightarrow \cdots
\end{array}$$

The upper row is the long exact sequence of the pair $(D(E), S(E))$. The map p^* is isomorphism because p is a homotopy equivalence. The *Euler class* $e \in H^n(B)$ is defined to be $(p^*)^{-1} j^*(u)$, so that the square containing $\smile e$ commutes. The other arrows in the lower row, too, are defined so that each square commutes. Then the lower row, which is forced to be exact, is called the *Gysin sequence* of the vector bundle.

The Euler class is, thus, the restriction of a Thom class to the zero section. As we recall, there is an map $H^n(D(E), S(E); \mathbb{Z}) \to H^n(D^n_x, S^{n-1}_x; \mathbb{Z})$ which is an isomorphism for oriented bundles for each $x \in B$, and takes a Thom class to the preferred choices of orientation of the fibres. In this view, a Thom class gives an orientation, and is specified by an orientation in turn. Thus, the Euler class depends only on the choice of orientation on the vector bundle. Obviously, a change in orientation only changes the sign of the Euler class.

2.4.1 Cohomology Ring of Infinite Grassmannians

Theorem 2.4.1. *$H^*(G_n, \mathbb{Z}/2)$ is a polynomial algebra in the indeterminates $w_1(E_n)$, $\cdots$,$w_n(E_n)$ over $\mathbb{Z}/2$; and $H^*(G_n(\mathbb{C}^\infty); \mathbb{Z})$ is a polynomial algebra in the indeterminates $c_1(E_n(\mathbb{C}^\infty))$,$\cdots$,$c_n(E_n(\mathbb{C}^\infty))$ over $\mathbb{Z}$. Multiplication is always given by cup product.*

Proof. We start by observing that the sphere bundle $S(E_n)$ naturally projects onto G_{n-1}. The projection $p : S(E_n) \to G_{n-1}$ sends $(l, v) \in S(E_n)$ to the $(n-1)$ plane in l orthogonal to v. This is a fibre bundle with fibre S^∞, which has the inductive topology $S^\infty = \cup_n S^n$. Note that, S^{n-1} is contractible in S^n, so we can contract S^∞ by contracting S^{n-1} in S^n in time $[1-2^{-n+1}, 1-2^{-n}]$. Thus, p induces isomorphism on all cohomology groups. By this isomorphism, the Gysin sequence of $E_n \to G_n$

with $\mathbb{Z}/2$ coefficient becomes

$$\cdots \to H^i(G_n) \xrightarrow{\smile e} H^{i+n}(G_n) \xrightarrow{\eta} H^{i+n}(G_{n-1}) \to H^{i+1}(G_n) \to \cdots$$

We want to first show that $\eta(w_j(E_n)) = w_j(E_{n-1})$. If we denote $\pi : S(E_n) \to G_n$, then η is $(p^*)^{-1}\pi^*$. The pullback bundle $\pi^*(E_n)$ consists of triples (l, w, v) where $w \in l$, and v is a unit vector in l; it splits naturally as a direct sum $L \oplus p^*(E_{n-1})$, where L is the line subbundle of triples (l, tv, v) with $t \in \mathbb{R}$ and v unit vector in l. L has a section $(l, v) \to (l, v, v)$, and hence is trivial. Thus, π^* takes $w_j(E_n)$ to $w_j(L \oplus p^*(E_{n-1})) = w_j(p^*(E_{n-1})) = p^*(w_j(E_{n-1}))$, so that $\eta(w_j(E_n)) = w_j(E_{n-1})$.

Now, we assume inductively that $H^*(G_{n-1})$ is the polynomial algebra on the classes $w_j(E_{n-1})$ for $j < n$. The beginning of the induction can be done with $n = 1$, since $w_1(E_1)$ is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$. Then the map η is surjective since $\eta(w_j(E_n)) = w_j(E_{n-1})$; so the Gysin sequence splits into short exact sequences:

$$0 \to H^i(G_n) \xrightarrow{\smile e} H^{i+n}(G_n) \xrightarrow{\eta} H^{i+n}(G_{n-1}) \to 0.$$

Now the image of $\smile e : H^0(G_n) \to H^n(G_n)$ is a copy of $\mathbb{Z}/2$ generated by e, which has to be the kernel of $\eta : H^n(G_n) \to H^n(G_{n-1})$. The class $w_n(E_n)$ lies in the kernel since $w_n(E_{n-1}) = 0$; but $w_n(E_n) \neq 0$. So, $w_n(E_n) = e$.

We have to prove that each $\xi \in H^k(G_n)$ can be expressed as a unique polynomial in classes $w_i = w_i(E_n)$. This we will prove by induction on k. First, $\eta(\xi)$ is a unique polynomial f in the $w_i(E_{n-1})$'s by induction on n, so that $\xi - f(w_1, \ldots, w_{n-1})$ lies in the kernel of η. By exactness, this lies in the image of $\smile e$, but $w_n = e$. So, there is a unique $\zeta \in H^{k-n}(G_n)$ (uniqueness by injectivity of $\smile e$) such that $\xi - f(w_1, \ldots, w_{n-1}) = \zeta \smile w_n$. Note that the polynomial f is uniquely determined; and by induction, ζ is a polynomial g in w_j's which is also uniquely determined. Thus the expression $\xi = f(w_1, \ldots, w_{n-1}) + w_n g(w_1, \ldots, w_n)$ is also uniquely determined. This completes the proof for the real Grassmannians.

For the complex case, we have to consider the complex Grassmannian $G_n(\mathbb{C}^\infty)$, and Chern classes instead of Stiefel-Whitney classes. The same argument as above, with "complex" replaced by "real", and with $\mathbb{Z}$ coefficient always, goes well, except at one point: c_n and e (the Z Euler class) both are in $\ker \eta$, which is a copy of $\mathbb{Z}$, and e is the generator of that, so that $c_n = me$ for some $m \in \mathbb{Z}$. Consider the bundle $E \to \mathbb{C}\mathrm{P}^\infty$ that is the direct sum of n copies of the tautological line bundle, classified by, say $f : \mathbb{C}\mathrm{P}^\infty \to G_n(\mathbb{C}^\infty)$; then we have for α a generator of $H^2(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z}) \cong \mathbb{Z}$, $\alpha^n = c_n(E) = f^*(c_n) = mf^*(e)$. This is an equality in $H^{2n}(\mathbb{C}\mathrm{P}^\infty; \mathbb{Z})$ with α^n a generator, which shows that m has to be ± 1. This completes the proof of the theorem. □

2.4.2 The Euler Class

We have the following proposition:

Proposition 2.4.2. a) *An orientation of a vector bundle $E \to B$ induces an orientation on the pullback bundle f^*E for map $f : X \to B$, in such a way that $e(f^*E) = f^*(e(E))$.*
b) *Orientations of two bundles $E_1, E_2 \to B$ determine an orientation of $E_1 \oplus E_2$ in such a way that $e(E_1 \oplus E_2) = e(E_1) \smile e(E_2)$.*
c) *For an orientable real bundle E which has rank n, the coefficient homomorphism $H^n(B;\mathbb{Z}) \to H^n(B;\mathbb{Z}/2)$ carries $e(E)$ to $w_n(E)$; and for a complex vector bundle E of rank n, and a suitable orientation thereof, we have $e(E) = c_n(E) \in H^{2n}(B;\mathbb{Z})$.*
d) *$e(E) = -e(E)$, if the fibres of E have odd dimension.*
e) *$e(E) = 0$, if E has a nowhere vanishing section.*

Proof. a) The map $\tilde{f} : f^*E \to E$ is an isomorphism in each fibre, so that if u is a Thom class for E, $\tilde{f}^*(u)$ restricts to a generator for $H^n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ for each fibre $\mathbb{R}^n$ of f^*E, so that it is a Thom class of f^*E. Euler class is the restriction of Thom class to the zero section, so a) follows.
b) Here we denote for a vector bundle E its complement of zero section by E'. Define the subset $E_{(1)}$ of $E = E_1 \oplus E_2$ to consist of vectors which does not go to zero under $\pi_1 : E_1 \oplus E_2 \to E_1$ (the natural projection on to the first coordinate); similarly define $E_{(2)}$. Note that $E_{(1)} \cup E_{(2)} = E'$. Let the ranks of E_1, E_2 be m, n, respectively. Then consider the following commutative diagram, in which integral coefficient is understood, and the horizontal maps are induced by restriction to a fibre:

$$
\begin{array}{ccc}
H^m(E_1, E_1') \times H^n(E_2, E_2') & \longrightarrow & H^m(\mathbb{R}^m, \mathbb{R}^m - 0) \times H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \\
\downarrow {\scriptstyle (\pi_1^*, \pi_2^*)} & & \downarrow \\
H^m(E, E_{(1)}) \times H^n(E, E_{(2)}) & \longrightarrow & H^m(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - \mathbb{R}^n) \times H^n(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - \mathbb{R}^m) \\
\downarrow {\scriptstyle \smile} & & \downarrow {\scriptstyle \smile} \\
H^{m+n}(E, E') & \longrightarrow & H^{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} - 0)
\end{array}
$$

On the right side, the upper vertical map induces isomorphism in each coordinate, and the lower one takes generator times generator to generator. This is for each pair of fibres. So, if u_1, u_1 are the Thom classes of E_1, E_2, respectively, then $\pi_1^*(u_1) \smile \pi_2^*(u_2)$ is a Thom class of E. Passing to the zero section, b) is immediate.
c) These are already shown for the universal bundles, in the proof of Theorem 2.4.1. As Euler class is functorial by a), the same holds for any arbitrary bundles.

d) With the notation used in the beginning of the section, we note that $\Phi(e) = p^*(e) \smile u = j^*(u) \smile u = u \smile u$, which means, the Euler class goes to the square of the Thom class under the Thom isomorphism. Thus, if n is odd, the basic commutativity relation for cup product gives $u \smile u = -u \smile u$, so that $e(E) = -e(E)$.
e) A nowhere vanishing section of E gives rise to a section of $S(E)$. In the part of long exact sequence

$$H^n(D(E), S(E); \mathbb{Z}) \xrightarrow{j^*} H^n(D(E); \mathbb{Z}) \xrightarrow{i^*} H^n(S(E); \mathbb{Z}),$$

the map i^* becomes injective as soon as we have a section of $S(E)$, since $D(E) \to B$ induces isomorphism on cohomology. Thus, in that case, j^* is zero, so that $e(E) = 0$. □

2.4.3 Examples and Application

Example 2.1. Taking direct sum with a product bundle does not change the total Stiefel-Whitney class, by Whitney sum formula. For the tangent bundle TS^n, the direct sum with the normal bundle, which is a trivial bundle as S^n is orientable, produces the product bundle $\mathbb{R}^{n+1}$. Hence the total Stiefel-Whitney class of TS^n is 1, i.e. $w_i(TS^n) = 0$ for $i > 0$.

Example 2.2. There is no bundle on $\mathbb{R}\mathrm{P}^\infty$ whose sum with E_1 produces a trivial bundle. Since, if there were such, its total Stiefel-Whitney class would be $(1 + w_1(E_1))^{-1}$. Now, $w_1(E_1) = \alpha$ is a generator of $H^1(\mathbb{R}\mathrm{P}^\infty; \mathbb{Z}/2)$. So all powers of α are non-zero; but $(1+\alpha)^{-1} = 1 + \alpha + \alpha^2 + \cdots$. If there were such a bundle, it would have finite rank, a contradiction.

Example 2.3. We have a canonical line bundle on $\mathbb{R}\mathrm{P}^n$, called γ_n^1, which is the pullback of the canonical line bundle E_1 on $\mathbb{R}\mathrm{P}^\infty$ under the natural inclusion. Thus, the total Stiefel-Whitney class of γ_n^1 is given by $1 + \alpha$, where α is a generator of $H^1(\mathbb{R}\mathrm{P}^n; \mathbb{Z}/2)$.

Example 2.4. The line bundle γ_n^1 admits an orthogonal complement bundle $\gamma^\perp$, say, inside the product bundle $\mathbb{R}^{n+1}$. Then, by Whitney sum formula, $w(\gamma_n^1)w(\gamma^\perp) = 1$, so that $w(\gamma^\perp) = 1 + \alpha + \cdots + \alpha^n$.

Example 2.5. Observe that if $\pi : S^n \to \mathbb{R}\mathrm{P}^n$ is the projection map identifying antipodal points, then $T\pi : TS^n \to T(\mathbb{R}\mathrm{P}^n)$ identifies two tangent vectors (x, v) and $(-x, -v)$. The tangent bundle of S^n is formed by vectors $(x, v) \in S^n \times R^{n+1}$ where v is in the orthogonal complement of the line determined by x. Thus, $T(\mathbb{R}\mathrm{P}^n)$ can be identified with the set of all pairs $\{(x, v), (-x, -v)\}$ with $\langle x, x\rangle = 1$ and $\langle x, v\rangle = 0$. Let L be a line through the origin in $\mathbb{R}^{n+1}$, intersecting S^n in $\pm x$; and let $L^\perp$ be the orthogonal complement space, then any such pairs as is mentioned in the preceding sentence, determines, and is determined by, a linear mapping $L \to L^\perp : x \mapsto v$. Thus, the tangent space of $\mathbb{R}\mathrm{P}^n$ at L is canonically identified with the vector space $\mathrm{Hom}(L, L^\perp)$, and the tangent bundle τ of $\mathbb{R}\mathrm{P}^n$ is canonically isomorphic with $\mathrm{Hom}(\gamma_n^1, \gamma^\perp)$.

Example 2.6. We still denote the tangent bundle of $\mathbb{R}\mathrm{P}^n$ by τ. γ_n^1 being a line bundle, $\mathrm{Hom}(\gamma_n^1, \gamma_n^1)$ is a trivial line bundle. If we write ϵ for the trivial line bundle on $\mathbb{R}\mathrm{P}^n$, then

$$\tau \oplus \epsilon \simeq \mathrm{Hom}(\gamma_n^1, \gamma^\perp) \oplus \mathrm{Hom}(\gamma_n^1, \gamma_n^1) = \mathrm{Hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) = \mathrm{Hom}(\gamma_n^1, \epsilon^{\oplus n}),$$

which is, again, $\mathrm{Hom}(\gamma_n^1, \epsilon)^{\oplus(n+1)} \simeq (\gamma_n^1)^{\oplus(n+1)}$. Hence,

$$w(\tau) = w(\tau \oplus \epsilon) = w(\gamma_n^1)^{n+1} = (1 + \alpha)^{n+1}.$$

Proposition 2.4.3. *If X has the homotopy type of a CW complex, then a vector bundle $E \to X$ is orientable if and only if $w_1(E) = 0$.*

Proof. We can assume that X itself is a CW complex; and by restricting to path components, we may further suppose that X is connected. There are natural isomorphisms

$$H^1(X; \mathbb{Z}/2) \xrightarrow{\simeq} \mathrm{Hom}(H_1(X), \mathbb{Z}/2) \xrightarrow{\simeq} \mathrm{Hom}(\pi_1(X), \mathbb{Z}/2),$$

the first one by universal coefficient theorem and the second, by the fact the $H_1(X)$ is the abelianization of $\pi_1(X)$. This takes $w_1(E_n)$ to an identification of $\pi_1(G_n)$ with $\mathbb{Z}/2$. By naturality, if there is a map $f : X \to G_n$, then $f^*(w_1(E_n))$ goes to the homomorphism $f_* : \pi_1(X) \to \pi_1(G_n) = \mathbb{Z}/2$. Hence $w_1(E) = f^*(w_1(E_n))$ is zero if and only if this homomorphism $f_* : \pi_1(X) \to \pi_1(G_n)$ is zero, which is exactly the condition of lifting f to the universal cover $\tilde{G}_n$, i.e., orientability of E. Note that the lifting criterion has a condition that the domain space has to be locally path connected, which is satisfied for CW complexes. □

Example 2.7. By the above, any vector bundle over a simply connected, CW complex base is orientable. Thus, TS^n is orientable for $n > 1$; for $n = 1$, it is trivial. For n odd, $e(TS^n) = 0$ by d) of Proposition 2.4.2. We calculate below the Euler class of TS^n when n is even:

Proposition 2.4.4. *For even n, $e(TS^n)$ is twice a generator of $H^n(S^n; \mathbb{Z})$.*

Proof. Let $E' \subset E = TS^n$ be the complement of zero section. Under Thom isomorphism, the Euler class $e(TS^n)$ corresponds to the square of the Thom class $c \in H^n(E, E'; \mathbb{Z})$; so, it suffices to show that c^2 is twice a generator of $H^{2n}(E, E'; \mathbb{Z})$. Let $A \subset S^n \times S^n$ consist of the antipodal pairs $(x, -x)$; then consider the map $f : S^n \times S^n - A \to E$ sending a pair (x, y) to the vector from x to the point of intersection of the line through $-x$ and y with the tangent space of S^n at x. It is a homeomorphism onto E. The diagonal D in $S^n \times S^n$ corresponds to the zero section of E. We have following isomorphisms, with integral coefficient understood:

$$H^*(E, E') \simeq H^*(S^n \times S^n, S^n \times S^n - D) \simeq H^*(S^n \times S^n, A) \simeq H^*(S^n \times S^n, D).$$

The first is by excision; second is by the deformation retract of $S^n \times S^n - D$ onto A by sliding (x, y) through the great circle through $x, -x, y$ towards $-x$; and the last is by the automorphism $(x, y) \to (x, -y)$ of $S^n \times S^n$ interchanging D and A. From the long exact sequence of the pair $(S^n \times S^n, D)$, we extract a short exact sequence

$$0 \to H^n(S^n \times S^n, D) \to H^n(S^n \times S^n) \to H^n(D) \to 0.$$

The middle group has generators α, β which are pullbacks of a generator of $H^n(S^n; \mathbb{Z})$ under two projections $S^n \times S^n \to S^n$. Both α and β restrict to the same generator of $H^n(D; \mathbb{Z})$, since the two projections restrict to the same homeomorphism $D \simeq S^n$. So, $(\alpha - \beta)$ generates $H^n(S^n \times S^n, D; \mathbb{Z})$. Now that E is orientable, either $(\alpha - \beta)$ or its inverse is a Thom class. See that $(\alpha - \beta)^2 = \alpha\beta - \beta\alpha$, which equals $-2\alpha\beta$, when n is even. But, $\alpha\beta$ is a generator of $H^{2n}(S^n \times S^n; \mathbb{Z}) \simeq H^{2n}(S^n \times S^n, D; \mathbb{Z})$. □

Chapter 3

Curvature and Chern Class

3.1 Connections and Curvature

In the following, X will denote a paracompact, smooth manifold, and $p : E \to X$ a complex vector bundle over X. The sheaf (of rings) of smooth, complex valued, functions on X will be denoted by $\mathscr{O}_X$, and the locally free sheaf (of $\mathscr{O}_X$-modules) of the smooth sections of E will be denoted by $\mathscr{E}$. The sheaf of sections of the complexified real tangent bundle of X will be denoted by $\mathscr{T}$, so that its complex dual will be written as $\mathscr{T}^*$, the sheaf of sections of the complex cotangent bundle. A *connection* on the bundle E is a morphism of sheaves of complex vector spaces (i.e. $\mathbb{C}$-linear) $\nabla : \mathscr{E} \to \mathscr{T}^* \otimes_{\mathscr{O}_X} \mathscr{E}$, which satisfies the Leibniz formula

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla(s),$$

for a smooth function f and a section s of E. It induces a map $\nabla^1 : \mathscr{T}^* \otimes_{\mathscr{O}_X} \mathscr{E} \to \Lambda^2(\mathscr{T}^*) \otimes_{\mathscr{O}_X} \mathscr{E}$ by $\nabla^1(\theta \otimes s) = \mathrm{d}\theta \otimes s - \theta \wedge \nabla(s)$ for a section s of E and a 1-form θ. The composition $K = \nabla^1 \circ \nabla : \mathscr{E} \to \mathscr{T}^* \otimes_{\mathscr{O}_X} \mathscr{E}$ is seen to be, actually, $\mathscr{O}_X$-linear, in the following way: let f be a smooth function on the manifold, and s a section of E, then

$$K(fs) = \nabla^1(\mathrm{d}f \otimes s + f\nabla s) = -\mathrm{d}f \wedge \nabla(s) + \mathrm{d}f \wedge \nabla(s) + f\nabla^1(\nabla(s)) = fK(s).$$

So, the map K can be interpreted as a global section of the sheaf of $\mathscr{O}_X$-modules $\underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{E}, \Lambda^2(\mathscr{T}^*) \otimes_{\mathscr{O}_X} \mathscr{E}) = \Lambda^2(\mathscr{T}^*) \otimes_{\mathscr{O}_X} \underline{\mathrm{Hom}}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{E})$. K, which now turns out to be an $\mathrm{End}(E)$-valued 2-form on X, is called the *curvature* of the connection ∇.

If we denote the real tangent bundle by $\mathscr{T}_{\mathbb{R}}$, and the sheaf of real valued smooth function by $\mathscr{S}$, then for a real vector bundle E with sheaf of smooth sections

$\mathscr{E}$, we can define a real connection as a map of sheaves of real vector spaces $\mathscr{E} \to \mathscr{T}^*_{\mathbb{R}} \otimes_{\mathscr{S}} \mathscr{E}$ which satisfies Leibniz formula. The curvature is defined similarly, and turns out to be an End(E)-valued 2-form on X.

A connection is completely determined by its local description as follows: if in a trivializing cover of E, there are n linearly independent sections $s_1, \ldots, s_n$ of E which span the fibres, then the equations $\nabla(s_i) = \sum_j \theta_{ij} \otimes s_j$ determine n^2 smooth 1-forms θ_{ij}, which, in turn, determine the connection ∇ uniquely in that trivializing cover.

If ∇_0, ∇_1 are connections on E, and ϕ a smooth function on X, then the linear combination $\phi \cdot \nabla_0 + (1 - \phi)\nabla_1$ is also a connection on E.

3.1.1 Existence of Connections

Let X be a paracompact, smooth manifold and E a vector bundle on X. The open subsets of X over which E is trivial constitute a cover of X, which by paracompactness has a locally finite refinement, say $\{V_\alpha\}_\alpha$; and there is a partition of unity $\{\phi_\alpha\}_\alpha$ subordinate to $\{V_\alpha\}_\alpha$. A trivial vector bundle admits connections. For, if it has a set of linearly independent global sections s_i, then if we choose any n global sections of $\mathscr{T}^* \otimes_{\mathscr{O}_X} \mathscr{E}$, say u_i, then we can define a connection $\nabla(\sum_i f_i s_i) = \sum_i (\mathrm{d}f_i \otimes s_i + f_i u_i)$. Once we have connections ∇_α on E_{V_α}, we can define a connection $\nabla = \sum_\alpha \phi_\alpha \nabla_\alpha$ on E. Hence,

Proposition 3.1.1. *On any vector bundle over a paracompact base, there is at least one connection.*

3.1.2 Pullback of Connection and Curvature

Given a map $f : Y \to X$ of smooth manifolds, the sheaf of sections of the pullback bundle f^*E is the pullback sheaf $f^*\mathscr{E}$, which is a locally free $\mathscr{O}_Y$-module. Locally, around a point $y \in Y$, a basis of $f^*\mathscr{E}$ is given by linearly independent sections of f^*E, which can chosen to be actually pullbacks of sections of E over a neighbourhood of $f(y) \in X$. If the local sections $s_1, \ldots, s_n$ around $f(y)$, as above, lift to sections $s'_1, \ldots, s'_n$ of f^*E, and span the fibres in a neighbourhood of y, then the pullback connection is completely determined by the equations $(f^*\nabla)s'_i = \sum_j f^*\theta_{ij} \otimes s'_j$, where $f^*\theta_{ij}$ is the pullback 1-form of θ_{ij}. For the pullback connection $f^*\nabla$, the curvature f^*K is called the pullback curvature.

Note that pullback of a connection is functorial, i.e., given maps $Z \xrightarrow{g} Y \xrightarrow{f} X$, and a vector bundle E over X, with a connection ∇, then $g^* f^* \nabla = (f \circ g)^* \nabla$. Same thing holds for curvature also.

3.1.3 The Connection Form

As we have seen in the foregoing, given a local frame, i.e. n-linearly independent local sections s_i which span the fibres in that neighbourhood, a connection produces a unique $n \times n$ matrix of 1-forms ω_{ij} defined by $\nabla(s_i) = \sum_j \omega_{ij} \otimes s_j$. The matrix $\omega = (\omega_{ij})$ is called the connection form with respect to the local frame s_i. Similarly, the curvature K also determines a matrix (Ω_{ij}) of 2-forms, called the curvature form, defined by $K(s_i) = \sum_j \Omega_{ij} \otimes s_j$.

We can relate Ω_{ij} with the connection form:

$$K(s_i) = \nabla^1(\sum_j \omega_{ij} \otimes s_j) = \sum_j (\mathrm{d}\omega_{ij} \otimes s_j - \omega_{ij} \wedge \nabla(s_j)) = \sum_k (\mathrm{d}\omega_{ik} - \sum_j \omega_{ij} \wedge \omega_{jk}) \otimes s_k,$$

which shows that

$$\Omega_{ik} = \mathrm{d}\omega_{ik} - \sum_j \omega_{ij} \wedge \omega_{jk}.$$

We can write it in a matrix form,

$$\Omega = \mathrm{d}\omega - \omega \wedge \omega.$$

If $f : Y \to X$ is a map, and if ∇ has the connection form ω_{ij} with respect to a local frame over an open set U, then the pullback connection $f^*\nabla$ has, with respect to the induced local frame over $f^{-1}(U)$, the connection form given by $f^*\omega_{ij}$. The above equation shows that, the curvature form is given by $f^*\Omega$.

With respect to a different local frame s'_i, with $s_i = \sum_{ij} a_{ij} s'_j$, the connection form ω' and the curvature form Ω' are given by the following equations:

$$\omega = \mathrm{d}a \cdot a^{-1} + a \cdot \omega' \cdot a^{-1}; \ \Omega = a \cdot \Omega' \cdot a^{-1},$$

where a is the matrix a_{ij} of smooth functions.

3.1.4 Connections Compatible with a Metric

Let E be a real vector bundle on X, and let h be a metric on E. A real connection ∇ is said to be compatible with h, if for any two sections s, s' of E we have

$$\mathrm{d}(h(s, s')) = h(\nabla s, s') + h(s, \nabla s').$$

Suppose that we have a connection ∇ and a metric h. Let $s_1, \ldots, s_n$ be a local orthonormal basis, with respect to which the connection from is ω_{ij}. Then we have

$$\begin{aligned} 0 = d(h(s_i, s_j)) &= h(\nabla s_i, s_j) + h(s_i, \nabla s_j) \\ &= h\Big(\sum \omega_{ik} \otimes s_k, s_j\Big) + \Big(s_i, \sum w_{jk} \otimes s_k\Big) = \omega_{ij} + \omega_{ji}. \end{aligned}$$

This means that the connection is compatible if and only if the connection form with respect to a local orthonormal frame is given by a *skew-symmetric* matrix.

3.1.5 Invariant Polynomials in The Curvature

$M_n(\mathbb{C})$ being identified with $\mathbb{C}^{n^2}$ in the natural way, one can talk about polynomial functions on $M_n(\mathbb{C})$, which are nothing but polynomials with complex coefficients in n^2 indeterminates. However, if for all complex matrices X, and nonsingular complex matrices T, such a polynomial P satisfies the relation $P(TXT^{-1}) = P(X)$, we will say that P is an *invariant polynomial*.

One notes that the subring of $\Lambda(\mathscr{T}^*)$ generated by even degree differential forms, is a commutative ring with unity. Suppose P is an invariant polynomial and K is the curvature for a given connection. With respect to some local frame, the curvature form is given by a matrix Ω. We will evaluate P at K in the following way: Ω being a matrix with entries in a commutative ring, it makes perfect sense to evaluate P at Ω; and as P is invariant, by the formulas in the last subsection, $P(\Omega) = P(\Omega')$. This defines the evaluation of P at K, which is independent of the local frame chosen. Written $P(K)$, it is an element of $\Lambda(\mathscr{T}^*)$; and in particular, if P is homogeneous of degree r, it is an exterior form of degree $2r$.

Lemma 3.1.2. *For an invariant polynomial P, the exterior form $P(K)$ is closed, i.e.,* $\mathrm{d}P(K) = 0$.

Proof. We will denote the polynomial obtained from $P((X_{ij})_{ij})$ by a formal derivative w.r.to X_{ij} by $\partial P/\partial X_{ij}$. The transpose of the matrix $(\partial P/\partial X_{ij})_{ij}$ will be denoted by P'. Let us fix a local frame, with respect to which the curvature form is the matrix $\Omega = (\Omega_{ij})$. The exterior derivative of $P(\Omega)$ is equal to $\sum_{ij} \partial P/\partial \Omega_{ij} \wedge \mathrm{d}\Omega_{ij}$, so that in matrix notation,

$$\mathrm{d}P(\Omega) = Tr(P'(\Omega) \wedge \mathrm{d}\Omega),$$

where Tr stands for trace.

If we take the exterior derivative of the matrix equation $\Omega = \mathrm{d}\omega - \omega \wedge \omega$, we get

$$\mathrm{d}\Omega = \omega \wedge \Omega - \Omega \wedge \omega,$$

which is called the *Bianchi identity*. Let E_{ij} denote the matrix whose ij-th element is 1 and other elements are zero. Then, for a matrix $A = (a_{ij})$, differentiating the expression

$$P((I + tE_{ij})A) = P(A(I + tE_{ij})),$$

with respect to t, and then setting $t = 0$, we get

$$\sum_{\beta} (\partial P/\partial a_{i\beta}) a_{j\beta} = \sum_{\alpha} (\partial P/\partial a_{\alpha j}) a_{\alpha i},$$

which means that the ji-th coefficient of the matrices $P'(A)A$ and $AP'(A)$ are equal. Since i, j are arbitrary, we infer that A commutes with $P'(A)$. The same will hold if we substitute Ω in place of A, since Ω has the coefficients in a commutative ring. So, we have the equation $\Omega \wedge P'(\Omega) = P'(\Omega) \wedge \Omega$. Writing ξ for the product matrix $P'(\Omega) \wedge \omega$, where ω is the connection form, we see that

$$\begin{aligned}
\mathrm{d}P(\Omega) &= Tr(P'(\Omega) \wedge d\Omega) \\
&= Tr(P'(\Omega) \wedge (\omega \wedge \Omega - \Omega \wedge \omega)) \quad \text{[Bianchi identity]} \\
&= Tr(\xi \wedge \Omega - \Omega \wedge \xi) \\
&= \sum_{ij} (\xi_{ij} \wedge \Omega_{ji} - \Omega_{ji} \wedge \xi_{ij}) \\
&= 0,
\end{aligned}$$

since Ω_{ji} being 2-forms, commute with all other forms, and in particular, with ξ_{ij}. □

So, given a connection and an invariant polynomial, we get an element in the de Rham cohomology ring. The following proposition shows that this class depends only on the polynomial P, not on the connection chosen:

Proposition 3.1.3. *The cohomology class determined by $P(K_\nabla)$, where K_∇ is the curvature of the connection ∇, is independent of the connection ∇.*

Proof. Let ∇_0 and ∇_1 be two connections on the vector bundle E on X. Then, the projection $\pi : X \times \mathbb{R} \to X$ defines two connections $\pi^*\nabla_0$ and $\pi^*\nabla_1$ on the pullback bundle π^*E. Define for $t \in [0, 1]$, the connection $\nabla = \phi \cdot \pi^*\nabla_1 + (1 - \phi) \cdot \pi^*\nabla_0$, where ϕ is the smooth function on $X \times \mathbb{R}$ that takes (x, t) to t. For each $t \in \mathbb{R}$, define the inclusion $i_t : X \to X \times \mathbb{R}$ by $i_t(x) = (x, t)$. This is a homotopy between

i_0 and i_1. So, the map $i_t^* : H_{dR}^*(X \times \mathbb{R}) \to H_{dR}^*(X)$ does not depend on t. But, by functoriality of pullback, $i_t^* \nabla$ is equal to $i_t^* \phi \cdot \nabla_1 + (1 - i_t^* \phi) \cdot \nabla_0 = t \cdot \nabla_1 + (1-t) \cdot \nabla_0$. Hence, $i_0^* \nabla = \nabla_0$ and $i_1^* \nabla = \nabla_1$, so that $i_0^* K\nabla = K_{\nabla_0}$ and $i_1^* K_\nabla = K_{\nabla_1}$. Since i_0^* is same as i_1^*, we conclude that

$$[P(K_{\nabla_0})] = [P(K_{\nabla_1})]$$

in the cohomology ring. □

This shows that the polynomial P determines a characteristic cohomology class $P(K) \in H^*(X; \mathbb{C})$ by de Rham isomorphism. For a map $f : Y \to X$, and a connection ∇ on a bundle $E \to X$, we have the pullback connection $f^*\nabla$ on f^*E, and the curvature $K_{f^*\nabla} = f^* K_\nabla$ by definition. It is evident that $P(K_{f^*\nabla}) = P(f^* K_\nabla) = f^*(P(K_\nabla))$, so that the cohomology class is also functorial.

3.2 The Chern Class in Terms of Curvature

If a complex line bundle E has a complex connection with matrix $[i\omega_{12}]$ and curvature matrix $[i\Omega_{12}]$, then the trace $Tr([i\Omega_{12}]) = i\Omega_{12}$ being an invariant polynomial, determines a characteristic (functorial) cohomology class in $H^2(X; \mathbb{C})$, which must be a multiple of its first Chern class $c_1(E) = e(E)$. If it is $ae(E)$, where a is a complex constant, then for determining a it is sufficient to calculate for a special case where the Chern class is non-zero. For that, we consider the unit sphere $\mathbb{C}\mathrm{P}^1$ and calculate both the classes for its cotangent bundle.

3.2.1 Complex Line Bundles on Riemannian Manifolds

If X is a manifold, and $\pi : E \to X$ is an oriented real vector bundle of rank 2, then there is an obvious almost complex structure J on E: having put a metric on E, if s_1, s_2 constitute a local orthonormal oriented basis of the bundle, then declare $Js_1 = s2$, $Js_2 = -s_1$. This makes sense, since, if e_1, e_2 is another oriented local orthonormal frame, then the transition matrix is in $SU(2)$, i.e. we can write $e_1 = cs_1 + ss_1$ and $e_2 = -ss_1 + cs_2$ for $c, s \in \mathbb{R}$ with $c^2 + s^2 = 1$. Then

$$Je_1 = cJ(s_1) + sJ(s_2) = cs_2 - ss_1 = e_2$$

and similarly, $J(e_2) = -e_1$. A compatible connection would give a skew symmetric matrix determined by, say, ω_{12} so that we can write

$$\nabla s_1 = \omega_{12} \otimes s_2; \quad \nabla s_2 = -\omega_{12} \otimes_{\mathbb{R}} s_1.$$

This evidently gives a complex connection, which we still write as ∇, given by $\nabla s_1 = \omega_{12} \otimes_{\mathbb{C}} s_2$. Writing $d\omega_{12} = \Omega_{12}$, we see that $[i\omega_{12}]$ is the connection matrix, with $[i\Omega_{12}]$ is the curvature matrix.

It is a well known fact that on the cotangent bundle of a Riemannian manifold, there is a unique connection compatible with the Riemannian metric. If the manifold is two dimensional, then to understand such a connection we have to compute only the term ω_{12} with respect to a single coordinate chart.

We introduce semigeodesic coordinates on the unit sphere. In the semigeodesic coordinates on a general Riemannian manifold, the matrix of the metric tensor looks like

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & G & \\ 0 & & & \end{pmatrix},$$

where G is an $(n-1) \times (n-1)$ matrix. For example, in the case of the sphere, if we take the cotangent bundle and introduce the coordinate system via latitude and longitude, then the metric quadratic differential would be $dx \otimes dx + \cos^2 x dy \otimes dy$. Take the orthonormal, oriented basis

$$\theta_1 = dx, \ \theta_2 = \cos x \mathrm{d}y.$$

Then the system of equations

$$\begin{aligned} \mathrm{d}\theta_1 &= \omega_{12} \wedge \theta_2 \\ \mathrm{d}\theta_2 &= -\omega_{12} \wedge \theta_1 \end{aligned}$$

have a unique solution $\omega_{12} = -\sin x \mathrm{d}y$. Thus, $\Omega_{12} = -\cos x \mathrm{d}x \wedge \mathrm{d}y$. With respect to the orientation determined by the outward normal, we can integrate this form, and we get

$$\iint \Omega_{12} = 4\pi.$$

However, the class $e(E)$ is twice the generator of $H^2(\mathbb{C}\mathrm{P}^1; \mathbb{Z})$, so, under the coefficient homomorphism

$$H^2(\mathbb{C}\mathrm{P}^1; \mathbb{Z}) \hookrightarrow H^2(\mathbb{C}\mathrm{P}^1; \mathbb{C}) = \mathbb{C}$$

it goes to 2. Hence, the Poincaré duality gives that the integral of $e(E)$ should be 2. This proves that

$$4\pi i = \iint i\Omega_{12} = a \iint e(E) = 2a,$$

which gives that $a = 2\pi i$.

What we have proved is the following:

Lemma 3.2.1. *Given a complex line bundle with a complex connection, its curvature determines a characteristic cohomology class, which is $2\pi i$ times its first Chern class.*

3.2.2 The Chern Class in Terms of Curvature

Now, let us have a complex vector bundle E of rank n with a complex connection ∇. We define an invariant polynomial for an $n \times n$ matrix A:

$$\underline{c}(A) = \det\Big(I + \frac{1}{2\pi i}A\Big) = \sum_k \sigma_k(A)/(2\pi i)^k.$$

Thus, for a line bundle with curvature K, $\underline{c}(K) = 1 + \frac{1}{2\pi i}K$. By the above lemma, it is equal to $1 + c_1(E) = c(E)$. We claim this equality for higher dimensional bundles:

Theorem 3.2.2. *Let $E \to X$ be a complex vector bundle with a connection ∇. Then $\underline{c}(K_\nabla) = c(E)$.*

Proof. To prove such equalities, we can assume that E splits as a direct sum of line bundles, by Lemma 2.2.3.

Then, if $E = \oplus_i E_i$, where E_i are line bundles, we can start from connections ∇_i on E_i, and define ∇ on E by the following rule: if ∇_i is given by the 1-form ω_i with respect to local frame of section s_i, then with respect to the local frame $(s_1, \ldots, s_n)$, ∇ is given by the matrix $\mathrm{diag}(\omega_1, \ldots, \omega_1)$. Thus, the curvature form is also given by $\mathrm{diag}(\Omega_1, \ldots, \Omega_n)$, where Ω_i are the curvature forms for ω_i. The polynomial $\underline{c}(K_\nabla)$ is evidently given by $\underline{c}(\Omega_1) \cdots \underline{c}(\Omega_n)$, which is equal to $c(E_1) \cdots c(E_n)$ by the above Lemma. But the last expression is equal to $c(E)$. □